A Reverse Hölder Type Inequality for the Logarithmic Mean and Generalizations

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Abstract

An inequality involving the logarithmic mean is established. Specifically, we show that

$$L(c,x)^{\frac{\ln(c/x)}{\ln(c/a)}}L(x,a)^{\frac{\ln(x/a)}{\ln(c/a)}} < L(c,a) \tag{1}$$

where $0 < a < x < c$ and $L(x,y) = \frac{y-x}{\ln y - \ln x}, \quad 0 < x < y$. Then several generalizations are given.

Key words Logarithmic mean, inequality

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1 Introduction

The logarithmic mean

$$L(y,x) = \frac{y-x}{\ln y - \ln x} \quad 0 < x < y,$$

has many applications in statistics and economics [8]. It is well known, and easily established [1,3,6,9]that

$$G(y,x) \leq L(y,x) \leq A(y,x)$$

where $G(y,x) = \sqrt{xy}$ **is the geometric mean and** $A(y,x) = (x+y)/2$ **is the arithmetic mean. In fact, writing** $A(y,x) = M_1(y,x)$ **where**

$$M_p(y,x) = \left(\frac{y^p + x^p}{2}\right)^{1/p}$$

it is known [6] that $M_{p1}(y,x) \leq M_{p2}(y,x)$ **for** $p1 \leq p2$ **It is also known, [4,5,8,11,13], that**

$$L(y,x) \leq M_{1/3}(y,x)$$

On the other hand, Hlder's inequality states that

$$M_1(y_1y_2, x_1x_2) \leq M_p(y_1,x_1)M_q(y_2,x_2)$$

if $1/p + 1/q = 1$ **with** $p,q > 0.$ **. It is thus curious that the logarithmic mean** $L(y,x)$ **satisfies the inequality**

$$L(c,x)^{\frac{\ln(c/x)}{\ln(c/a)}} L(x,a)^{\frac{\ln(x/a)}{\ln(c/a)}} < L(c,a) \tag{2}$$

where $0 < a < x < c$ **and it is noted that**

$$\frac{\ln(c/x)}{\ln(c/a)} + \frac{\ln(x/a)}{\ln(c/a)} = 1$$

It is the reverse Hölder type inequality (1) which is the subject of this note and will be established below. (1) arises in a parameter identification problem for a fractal Michaelis-Mention equation [7]. In the following, use will be made of Jensen's inequality [10] which we now state for the reader's convenience:

1.1. Jensen's Inequality
1). if $w_i > 0\ \ \forall\ i = 1, 2, \ldots, n$
2). $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \in R$
3). $\Phi : [0, \infty) \to R$ *is a strictly convex function*
then

$$\left(\sum_{i=1}^{n} w_i\right) \Phi\left(\frac{\sum_{i=1}^{n} w_i \alpha_i}{\sum_{i=1}^{n} w_i}\right) \leq \sum_{i=1}^{n} w_i \Phi(\alpha_i)$$

and the inequality is strict unless $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_n$.

2 Main Result

Lemma 2.1. *Let* $g(u) = \frac{nu}{u-1}$ *where* $g(1) = 1$*. Then* $\forall\ \ u > 0$
i). g is a strictly decreasing functin of u
ii).
iii). $\lim_{u \to 0^+} g(u) = \infty, \quad \lim_{u \to \infty} g(u) = 0, \quad \lim_{u \to 1} g(u) = 1$
iv). $g(1/u) = ug(u)$.

Proof:

Set $z(u) = 1 - 1/u - \ln u$ **then** $z'(u) = \frac{1}{u}\left(\frac{1}{u} - 1\right)$ **which is positive for** $0 < u < 1$ **and negative for** $u > 1$. **Thus** $z(u)$ **increase from** $-\infty$ **to** 0 **at** $u = 1$ **and then decreases to** $-\infty$ **as u tends to** ∞. **Thus** $g'(u)$ **is negative except at u = 1. This establishes** ***(i). The limits in (*ii) can be computed in the usual fashion using L'hopital's rule. For** ***(iii) we have***

$$g(1/u) = \frac{\ln(1/u)}{1/u - 1} = ug(u).$$

□

Lemma 2.2. *Let* $f(x) = x - \ln x$*, then*
i). f is decreasing on (0,1) and increasing on (1,∞)
ii). $\lim_{x \to 0^+} f(x) = \infty, \quad f(1) = 1,$ *and* $\lim_{x \to \infty} f(x) = \infty$
iii). if $\alpha > 0, \quad x > 0$ *then* $f(\alpha x) = f(x)$ *for* $x = g(\alpha)$ *so that* $f(\alpha g(\alpha)) = f(g(\alpha))$.
Proof:

(i) and* (ii) can be established in the usual way. For *(iii) we have

$$f(\alpha\, x) = f(x) \Rightarrow \alpha\, x - \ln(\alpha\, x) = x - \ln x \Rightarrow (\alpha - 1)x = \ln \alpha \Rightarrow x = g(\alpha).$$

□

Let $y(x)$ ***denote the left hand side of (1) and set*** $\alpha = \ln c - \ln a$. ***Note that*** $y(x) > 0\ \ \forall\ \ a < x < c$. ***Then***

$$\alpha \ln y = [\ln c - \ln x]\,[\ln{(c - x)} - \ln(\ln c - \ln x)] + [\ln x - \ln a]\,[\ln(x - a) - \ln(\ln x - \ln a)]$$

and so

$$\frac{\alpha\, y'}{y} = -\frac{1}{x}\left[\ln(c-x) - \ln(\ln c - \ln x)\right] + \left[\ln c - \ln x\right]\left[\frac{-1}{c-x} - \frac{-1/x}{\ln c - \ln x}\right] +$$
$$\frac{1}{x}\left[\ln(x-a) - \ln(\ln x - \ln a)\right] + \left[\ln x - \ln a\right]\left[\frac{1}{x-a} - \frac{1/x}{\ln x - \ln a}\right]$$
$$= \frac{1}{x}\left[\ln\left[\frac{x-a}{\ln x - \ln a}\right]\right] + \left[\frac{\ln x - \ln a}{x-a}\right] - \frac{1}{x} + \frac{1}{x} - \frac{\ln c - \ln x}{c-x} - \frac{1}{x}\ln\left[\frac{c-x}{\ln c - \ln x}\right]$$
$$= \frac{1}{x}\ln\left[x\frac{a/x-1}{\ln(a/x)}\right] + \frac{1}{x}\frac{\ln(a/x)}{a/x-1} - \frac{1}{x}\frac{\ln(c/x)}{c/x-1} - \frac{1}{x}\ln\left[x\frac{c/x-1}{\ln(c/x)}\right] \tag{3}$$

or

$$\frac{\alpha\, y'}{y} = \frac{1}{x}\left[f(g(a/x)) - f(g(c/x))\right] \;=\; \frac{1}{x}h(x) \tag{4}$$

Now $f(g(a/x))$ ***is an increasing function of x while*** $f(g(c/x))$ ***is a decreasing function of x so that*** $h(x)$ ***is an increasing function of x. Clearly*** $\alpha y'/y$ ***is zero at exactly one point which implies that*** y' ***is zero at exactly one point.***

Lemma 2.3. *y' is zero at the point $x = \sqrt{ac}$.*
Proof:
Now $f(g(c/x)) = f(g(a/x)) = f\left(\frac{a}{x}g(a/x)\right)$ **, from lemma 2.3** ***(iii), so that*** $g(c/x) = (a/x)g(a/x) = g(x/a)$ ***by lemma 2.2*** **(iii). Thus** $c/x = x/a$ **giving** $x = \sqrt{ac}$**.** □

Theorem 2.1. *For all values of* $0 < a < x < c$

$$\left(\frac{c-x}{\ln c - \ln x}\right)^{\ln c - \ln x}\left(\frac{x-a}{\ln x - \ln a}\right)^{\ln x - \ln a} < \left(\frac{c-a}{\ln c - \ln a}\right)^{\ln c - \ln a} \tag{5}$$

Proof:
The results hold iff

$$(\ln c - \ln x)\left(\frac{c-x}{\ln c - \ln x}\right) + (\ln x - \ln a)\left(\frac{x-a}{\ln x - \ln a}\right) < (\ln c - \ln a)\left(\frac{c-a}{\ln c - \ln a}\right)$$

Set $x_0 = a,\ x_1 = x,\ x_2 = c$ **and let** $w_i = \ln x_i - \ln x_{i-1},\ \alpha_i = \frac{x_i - x_{i-1}}{\ln x_i - \ln x_{i-1}}$ **and let** $\Phi\,(x) = -\ln x$**, the result follows from the Jensen's inequality with** $\leq$ **rather than** $<$**.**
But

$$\alpha\, y' = \frac{y}{x}\left[f(g(a/x)) - f(g(c/x))\right]$$

so that y' **is negative on** $[a, \sqrt{ac}]$ **and positive on** $[\sqrt{ac}, c]$**. Strict inequality in Theorem 2.4 now follows from the previous results since the derivative is strictly negative on** $[a, \sqrt{ac}]$ **and positive on the interval** $[\sqrt{ac}, c]$**. Thus equality holds only at a and c.** □

3 Convexity

Theorem 3.1. *The function*

$$y(x)=\left(\frac{c-x}{\ln c-\ln x}\right)^{\frac{\ln c-\ln x}{\ln c-\ln a}}\left(\frac{x-a}{\ln x-\ln a}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}} \tag{6}$$

is log-convex, and hence convex, on the interval $\sqrt{ac}$.

Proof:
Let $w=\alpha\ln y$**, then** $w'=\alpha y'/y$ **and hence from (5)** $xw'=f(g(a/x))-f(g(c/x))$ **is an increasing function so that** $w'+xw''\geq 0$. **Thus** $xw''\geq -w'$. **Now on** $[a,\sqrt{ac}]$, $w'\leq 0$, **and so** $w''\geq 0$ **so that** w **is convex (and hence log convex) on** $[a,\sqrt{ac}]$. $\Box$

Lemma 3.1. *The curve*

$$y(x)=\left(\frac{c-x}{\ln c-\ln x}\right)^{\frac{\ln c-\ln x}{\ln c-\ln a}}\left(\frac{x-a}{\ln x-\ln a}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}} \tag{6}$$

is invariant under the transformation $x\leftarrow ac/x$.

Proof:

$$\begin{aligned}
z(x) &= \left(\frac{c-\frac{ac}{x}}{\ln c-\ln\left(\frac{ac}{x}\right)}\right)^{\frac{\ln c-\ln\left(\frac{ac}{x}\right)}{\ln c-\ln a}}\left(\frac{\frac{ac}{x}-a}{\ln\left(\frac{ac}{x}\right)-\ln a}\right)^{\frac{\ln\left(\frac{ac}{x}\right)-\ln a}{\ln c-\ln a}}\\
&= \left(\frac{\frac{c(x-a)}{x}}{\ln c-\ln a-\ln c+\ln x}\right)^{\frac{\ln c-\ln(ac)+\ln x}{\ln c-\ln a}}\left(\frac{\frac{a(c-x)}{x}}{\ln(ac)-\ln x-\ln a}\right)^{\frac{\ln(ac)-\ln x-\ln a}{\ln c-\ln a}}\\
&= \left(\frac{\frac{c(x-a)}{x}}{\ln x-\ln a}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}}\left(\frac{\frac{a(c-x)}{x}}{\ln c-\ln x}\right)^{\frac{\ln c-\ln x}{\ln c-\ln a}}\\
&= \left(\frac{\mathbf{c}}{\mathbf{x}}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}}\left(\frac{a}{x}\right)^{\frac{\ln c-\ln x}{\ln c-\ln a}}y(x).
\end{aligned} \tag{7}$$

Now

$$\frac{\ln x-\ln a}{\ln c-\ln a}+\frac{\ln c-\ln x}{\ln c-\ln a}=1$$

Thus from (7)

$$\begin{aligned}
\left(\frac{c}{x}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}}\left(\frac{a}{x}\right)^{\frac{\ln c-\ln x}{\ln c-\ln a}} &= \left(\frac{c}{x}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}}\left(\frac{a}{x}\right)^{1-\frac{\ln x-\ln a}{\ln c-\ln a}}\\
&= \frac{\left(\frac{c}{x}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}}}{\left(\frac{a}{x}\right)^{\frac{\ln x-\ln a}{\ln c-\ln a}}}\frac{a}{x}\\
&= \left(\frac{a}{x}\right)\left(\frac{c}{a}\right)^{\ln(x/a)/\ln(c/a)}\\
&= \frac{a}{x}\frac{x}{a}=1 \text{ since } b^x=e^{x\ln b}.
\end{aligned}$$

Thus z(x) = y(x) and the lemma is proved. $\square$

4 Generalizations and Applications

The following theorems follow directly from Jensen's inequality and are generalizations of Theorem 2.1.

Theorem 4.1. *if:*
i). $\Phi : [0, \infty) \to R$ *is a function*
ii). $f, g : [0.\infty) \to R$ *are increasing functions*
3). $A_0, A_1, \ldots, A_n$
then
1). If Φ *is convex*

$$(g(A_n) - g(A_0))\ \Phi\left(\frac{f(A_n) - f(A_0)}{g(A_n) - g(A_0)}\right) \leq \sum_{i=1}^{n} (g(A_i) - g(A_{i-1}))\ \Phi\left(\frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}\right)$$

2). If Φ *is concave then*

$$(g(A_n) - g(A_0))\ \Phi\left(\frac{f(A_n) - f(A_0)}{g(A_n) - g(A_0)}\right) \geq \sum_{i=1}^{n} (g(A_i) - g(A_{i-1}))\ \Phi\left(\frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}\right)$$

3). If Φ *is log convex then*

$$\Phi\left(\frac{f(A_n) - f(A_0)}{g(A_n) - g(A_0)}\right)^{(g(A_n) - g(A_0))} \leq \prod_{i=1}^{n} \Phi\left(\frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}\right)^{(g(A_i) - g(A_{i-1}))}$$

4). If Φ *log concave then*

$$\Phi\left(\frac{f(A_n) - f(A_0)}{g(A_n) - g(A_0)}\right)^{(g(A_n) - g(A_0))} \geq \prod_{i=1}^{n} \Phi\left(\frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}\right)^{(g(A_i) - g(A_{i-1}))}$$

Proof:
In Jensen's inequality set $w_i = g(A_i) - g(A_{i-1})$ **and** $\alpha_i = \frac{f(A_i) - f(A_{i-1})}{g(A_i) - g(A_{i-1})}$ **and the result follows.** $\square$

As a first application let $M, N : R \to R$ **N strictly monotone. Given any two numbers a and b, there is a number c, according to the mean value theorem, such that**

$$\frac{M(b) - M(a)}{N(b) - N(a)} = \frac{M'(c)}{N'(c)}$$

for some c, $a < c < b$. **If c is uniquely determined then it is called the (M.N) mean-value mean of a and b [2]. In this case let H be the inverse of** M'/N' **and write**

$$c = H\left(\frac{M(b) - M(a)}{N(b) - N(a)}\right)$$

If M and N are both increasing and H is either log-convex or log-concave, we can apply one of the inequalities in Theorem 4.1 to write

$$H\left(\frac{M(A_n)-M(A_0)}{N(A_n)-N(A_0)}\right) \leq \prod_{i=1}^{n} H\left(\frac{M(A_i)-M(A_{i-1})}{N(A_i)-N(A_{i-1})}\right)^{\frac{N(A_i)-N(A_{i-1})}{N(A_n)-N(A_0)}}$$

or

$$H\left(\frac{M(A_n)-M(A_0)}{N(A_n)-N(A_0)}\right) \geq \prod_{i=1}^{n} H\left(\frac{M(A_i)-M(A_{i-1})}{N(A_i)-N(A_{i-1})}\right)^{\frac{N(A_i)-N(A_{i-1})}{N(A_n)-N(A_0)}}$$

where we have made the associations that $\Phi = H,\ f = M,\ g = N,\ A_n = b,\ A_0 = a$

Now specializing to the case of $\Phi\ (x)$ **(log-concave** Φ **) in theorem 4.1 we obtain**

$$\frac{f(A_n)-f(A_0)}{g(A_n)-g(A_0)} \geq \prod_{i=1}^{n}\left(\frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}\right)^{\frac{g(A_i)-g(A_{i-1})}{g(A_n)-g(A_0)}}$$

and interchanging f and g we can write

$$\frac{f(A_n)-f(A_0)}{g(A_n)-g(A_0)} \leq \prod_{i=1}^{n}\left(\frac{f(A_i)-f(A_{i-1})}{g(A_i)-g(A_{i-1})}\right)^{\frac{f(A_i)-f(A_{i-1})}{f(A_n)-f(A_0)}}$$

From these expressions we can obtain inequalities for Stolarsky's ([2], [12]) extended mean value

$$E_{r,s}(a,b) = \left(\frac{r\,(a^s-b^s)}{s\,(a^r-b^r)}\right)^{\frac{1}{s-r}}$$

by making the associations $f(x) = x^s/s,\ g(x) = x^r/r,\ A_n = b,\ A_0 = a$ **and then raising both sides to the power 1/(s-r). For** $rs > 0$

$$\left(\frac{b^s-u^s}{b^r-u^r}\right)^{\frac{b^r-u^r}{b^r-a^r}}\left(\frac{u^s-a^s}{u^r-a^r}\right)^{\frac{u^r-a^r}{b^r-a^r}} \leq \frac{b^s-a^s}{b^r-a^r} \leq \left(\frac{b^s-u^s}{b^r-u^r}\right)^{\frac{b^s-u^s}{b^s-a^s}}\left(\frac{u^s-a^s}{u^r-a^r}\right)^{\frac{u^s-a^s}{b^s-a^s}}$$

where $a < u < b$.

If $rs < 0,\ f(x) = x^s/s$ **and** $g(x) = x^r/r$ **are still both increasing functions and we have a similar inequality**

$$\left(\frac{r\,(b^s-u^s)}{s\,(u^r-a^r)}\right)^{\frac{b^r-u^r}{b^r-a^r}}\left(\frac{r\,(u^s-a^s)}{s\,(u^r-a^r)}\right)^{\frac{u^r-a^r}{b^r-a^r}} \leq \frac{r\,(b^s-a^s)}{s\,(b^r-a^r)} \leq \left(\frac{r\,(b^s-u^s)}{s\,(b^r-u^r)}\right)^{\frac{b^s-u^s}{b^s-a^s}}\left(\frac{r\,(u^s-a^s)}{s\,(u^r-a^r)}\right)^{\frac{u^s-a^s}{b^s-a^s}}$$

where it is now necessary to include r/s or else reverse the inequality.

A further application is obtained by setting $f(x) = x$ and $g(x) = \ln x$ above to obtain

$$\left(\frac{A_n - A_0}{\ln(A_n) - \ln(A_0)}\right)^{\ln(A_n)-\ln(A_0)} \geq \prod_{i=1}^{n}\left(\frac{A_i - A_{i-1}}{\ln(A_i) - \ln(A_{i-1})}\right)^{\ln(A_i)-\ln(A_{i-1})}$$

and

$$\left(\frac{A_n - A_0}{\ln(A_n) - \ln(A_0)}\right)^{A_n-A_0} \leq \prod_{i=1}^{n}\left(\frac{A_i - A_{i-1}}{\ln(A_i) - \ln(A_{i-1})}\right)^{A_i-A_{i-1}}$$

These two inequalities provide a direct generalization and converse to the main inequality (3) discussed in this paper

$$\left(\frac{c-x}{\ln c - \ln x}\right)^{\ln c - \ln x}\left(\frac{x-a}{\ln x - \ln a}\right)^{\ln x - \ln a} < \left(\frac{c-a}{\ln c - \ln a}\right)^{\ln c - \ln a}.$$

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