Nonlinearity **12** (1999) 617–633. Printed in the UK PII: S0951-7715(99)97552-7

Nonchaotic behaviour in three-dimensional quadratic systems II. The conservative case

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Received 15 September 1998, in final form 26 January 1999
Recommended by E B Bogomolny

Abstract. It is shown that almost all three-dimensional conservative quadratic systems of ordinary differential equations with a total of four terms on the right-hand side of the equations do not exhibit chaos. A previous paper showed the same thing for dissipative systems.

AMS classification scheme numbers: 34C99, 34045
PACS number: 0545

1. Introduction

The authors have recently shown [7] that three-dimensional dissipative quadratic systems of ordinary differential equations with a total of only four terms on the right-hand side do not exhibit chaos. This complements recent papers of Sprott [5, 6] who has given numerous examples of chaotic three-dimensional quadratic systems with as few as five terms on the right-hand side, only one of which need be quadratic. We believe that all four-term three-dimensional quadratic systems (dissipative or not) are nonchaotic. The problem is that there are a huge number, many hundreds, of different possibilities. A brief discussion of this complexity was given in the last section of [7].

In order to make further progress on this problem, the next-simplest category, after dissipative systems, is conservative systems. Indeed, Sprott [6] has also found a chaotic example of a conservative five-term three-dimensional system with only one nonlinear term (see also [1]). Again, complementing Sprott's work, we show in this paper (with one exception described below) that all conservative three-dimensional quadratic systems with a total of four terms on the right-hand side are nonchaotic.

The general method is the same as in the previous paper. As before, it turns out that all such systems (again, with one exception, described below) which are neither integrable, nor reducible to a two-dimensional system, nor essentially linear, can have only two types of behaviour. Solutions are either asymptotic to a two-dimensional surface or at least one component has an infinite limit. In the previous paper if one component had an infinite limit, the other two components were also shown to have limits. Since the existence of a single component with an infinite limit implies the system is nonchaotic, we do not carry the argument

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0951-7715/99/030617+17$19.50

any further in this paper. Of course, both types of behaviour may occur in a single equation. Initial conditions have no effect on our asymptotic methods and are therefore ignored.

There is no simple systematic method for picking out the conservative systems from the totality of all systems. As in the first paper, the tedious method of listing all possible systems was used and then picking out the class (conservative in this case) of particular equations desired.

As in the first paper, four-term systems can be rescaled to eliminate all parameters and this is automatically done for each equation, possibly leaving an arbitrary $\pm$ sign on one term. This rescaling may (or may not) reverse the time parameter t. If time is reversed then it can be re-reversed simply by multiplying each term on the right-hand side by (-1). Such a sign reversal does not affect the methods used in this paper and therefore does not affect the results obtained. Thus, this possible reversal of time will be ignored with no loss of generality.

It is a curious fact that the class of conservative systems, defined by $\nabla \cdot f = 0$, where $\dot{x} = f(x)$ is the system in vector form, is much larger than the class of dissipative systems defined by $\nabla \cdot f < 0$. Furthermore the geometric significance of the conservative property, that volumes in phase space are conserved along flows, plays no role in the analysis.

The plan of this paper is similar to our previous one. We take up, successively, equations with either one, two, three or four nonlinear terms, all without constant terms. Then systems with a constant term are considered separately. Since there are so many more specific conservative systems than dissipative ones, and since the methods are the same as in the previous paper, our treatment is much more abbreviated, with more details left to be filled in by the reader.

The one case which is not rigorously resolved is for the system $\dot{x} = y^2 - z^2$, $\dot{y} = x$, $\dot{z} = y$ which has the scalar form $\dddot{z} = \dot{z}^2 - z^2$. Here we show that any nonoscillatory solution is unbounded and that there appears, numerically, to be a unique oscillatory (in fact, periodic) solution.

2. Four-term systems with one quadratic term

The conservative systems with four terms and one quadratic term are:

$$\begin{cases} \dot{x} = y^2 + ky \\ \dot{y} = z \\ \dot{z} = x \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x \\ \dot{z} = y \end{cases} \tag{2.2}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = z \\ \dot{z} = x \end{cases} \tag{2.3}$$

$$\begin{cases} \dot{x} = yz + x \\ \dot{y} = -y \\ \dot{z} = x \end{cases} \tag{2.4}$$

$$\begin{cases} \dot{x} = yz + y \\ \dot{y} = x \\ \dot{z} = y \end{cases} \tag{2.5}$$

$$\begin{cases} \dot{x} = yz + y \\ \dot{y} = z \\ \dot{z} = x \end{cases} \tag{2.6}$$

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = x + z \\ \dot{z} = x \end{cases} \tag{2.7}$$

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = x + z \\ \dot{z} = y \end{cases} \tag{2.8}$$

$$\begin{cases} \dot{x} = yz \\ \dot{y} = x + z \\ \dot{z} = x \end{cases} \tag{2.9}$$

$$\begin{cases} \dot{x} = yz \\ \dot{y} = x + z \\ \dot{z} = y \end{cases} \tag{2.10}$$

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = z \\ \dot{z} = x + y \end{cases} \tag{2.11}$$

Theorem 1. *Systems (2.1)–(2.11) are nonchaotic.*

Proof. System (2.4) reduces to a two-dimensional linear system. Systems (2.5) and (2.10) reduce to the scalar equation $\dddot{z} - z\dot{z} - \dot{z} = 0$ which is integrable.

All of the remaining systems except (2.9) have scalar forms whose solutions are either asymptotic to a two-dimensional surface or have at least one component with an infinite limit as follows:

(2.1)	$\dddot{y} - y^2 - ky = 0$	(multiply by $\dot{y}$ and integrate),
(2.2)	$\dddot{z} - \dot{z}^2 - z = 0$	(multiply by $\ddot{z}$ and integrate),
(2.3)	$\dddot{y} - y^2 - \dot{y} = 0$	(multiply by $\dot{y}$ and integrate),
(2.6)	$\dddot{y} - y\dot{y} - y = 0$	(multiply by y and integrate),
(2.7)	$\dddot{y} - 2y\dot{y} - y^2 = 0$	(integrate),
(2.8)	$\dddot{z} - \dot{z}^2 - \dot{z} = 0$	(integrate),
(2.11)	$\dddot{y} - y^2 - \dot{y} = 0$	(integrate).

For system (2.9) the scalar form is $\dddot{z}z = \dot{z}z^2 + z^3 + \ddot{z}\dot{z}$. Dividing by z and integrating gives

$$\ddot{z} = \frac{1}{2}z^2 + \int_0^t z^2(s)\,\mathrm{d}s + \int \frac{\ddot{z}\dot{z}}{z} + c.$$

Note that $\frac{\ddot{z}}{z} = y$, $\dot{z} = x$ and $xy = y\dot{y} - \dot{x}$. Thus

$$\int \frac{\ddot{z}\dot{z}}{z} = \frac{y^2}{2} - x = \frac{1}{2}\left(\frac{\ddot{z}}{z}\right)^2 - \dot{z}$$

and hence we have

$$\ddot{z} - \frac{1}{2}z^2 - \frac{1}{2}\left(\frac{\ddot{z}}{z}\right)^2 + \dot{z} = c + \int_0^t z^2(s)\,\mathrm{d}s.$$

Thus the left-hand side above is asymptotic to a two-dimensional surface or else $\dot{z}(t) \to \infty$ as $t \to \infty$. □

3. Four-term systems with two quadratic terms

The conservative four-term systems with two quadratic terms are:

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x \\ \dot{z} = y \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = z \\ \dot{z} = x \end{cases} \tag{3.2}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = x \\ \dot{z} = y \end{cases} \tag{3.3}$$

$$\begin{cases} \dot{x} = 2xy + z \\ \dot{y} = -y^2 \\ \dot{z} = x \end{cases} \tag{3.4}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = xz \\ \dot{z} = z \end{cases} \tag{3.5}$$

$$\begin{cases} \dot{x} = xz + y \\ \dot{y} = -yz \\ \dot{z} = x \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{x} = x^2 + z \\ \dot{y} = -2xy \\ \dot{z} = y \end{cases} \tag{3.7}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = xz \\ \dot{z} = \pm y \end{cases} \tag{3.8}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = z^2 \\ \dot{z} = x \end{cases} \tag{3.9}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x^2 \\ \dot{z} = \pm x \end{cases} \tag{3.10}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x^2 \\ \dot{z} = \pm y \end{cases} \tag{3.11}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = \pm xz \\ \dot{z} = y \end{cases} \tag{3.12}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = z^2 \\ \dot{z} = \pm x \end{cases} \tag{3.13}$$

$$\begin{cases} \dot{x} = yz + x \\ \dot{y} = x^2 \\ \dot{z} = -z \end{cases} \tag{3.14}$$

Theorem 2. *Systems (3.1)–(3.3)+ and (3.4)–(3.13) are not chaotic.*

Proof. System (3.4) reduces to a two-dimensional linear system. (3.14) has the scalar form $\ddot{x} = x + ce^{-t}x^2$ which is equivalent to an Emden–Fowler equation [4, p 400]. The remaining systems all have scalar forms whose solutions are asymptotic to a two-dimensional surface or have at least one component with a limit as $t \to \infty$ as follows:

(3.1) $\dddot{z} - \dot{z}^2 - z\dot{z} = 0$ (integrate),

(3.2) $\dddot{y} - y^2 - y\dot{y} = 0$ (integrate),

(3.3) + $\dddot{z} - \dot{z}^2 - z^2 = 0$ (integrate),

(3.5) $\ddot{y} - ce^t y^2 = 0$ (integrate),

(3.6) $\dddot{z} - \dot{z}z^2 - \dot{z}^2 = 0$ (integrate),

(3.7) $\dddot{x} - 2\dot{x}^2 - 4x^2\dot{x} = 0$ (integrate),

(3.8) $\dddot{z}z \mp \dot{z}^2z^2 \mp z^2\dot{z} \mp \dot{z}\ddot{z} = 0$ (integrate),

(3.9), (3.10), (3.11), (3.13) $\Big\}$ y is monotone and thus has a limit as $t \to \infty$.

□

System (3.12) has the scalar form $z\dddot{z} \mp \dot{z}^2z^2 \mp z^3 - \ddot{z}\dot{z} = 0$. Dividing by z^3, multiplying by $\ddot{z}$, and integrating leads to

$$\frac{\ddot{z}^2}{2z^2} \mp \frac{\dot{z}^3}{3z} \mp \dot{z} = c + \int_{t_0}^{t} \frac{(\dot{z}(s))^4}{3(z(s))^2}\,\mathrm{d}s.$$

To proceed further we consider (3.12)− and (3.12)+ separately and begin with the following

Lemma 1. *The scalar form of (3.12)− has no oscillatory solutions.*

Proof. Suppose that $z(t_1) = z(t_3) = 0$ with $z(t) > 0$ for $t_1 < t < t_3$ and $\dot{z}(t_2) = 0$, $t_1 < t_2 < t_3$. Suppose first that $\dot{z}(t_1) > 0$ and write

$$\left.\left(\frac{\ddot{z}^2}{2z^2} + \frac{\dot{z}^3}{3z} + \dot{z}\right)\right|_{t_2} = \left.\left(\frac{\ddot{z}^2}{2z^2} + \frac{\dot{z}^3}{3z} + \dot{z}\right)\right|_{t} + \int_{t}^{t_2} \frac{(\dot{z}(s))^4}{3(z(s))^2}\,\mathrm{d}s, \qquad t_1 < t < t_2.$$

Then for the above expression, the left-hand side is finite while for $t \downarrow t_1$, the right-hand side $\to \infty$, a contradiction.

Now suppose that $\dot{z}(t_1) = 0$. Note that $x(t_2) = -\ddot{z}(t_2)/z(t_2) \geqslant 0$ and also that $\dot{x}(t) > 0$ for $t_1 < t < t_3$ (since $z > 0$ on this interval). Thus $x(t) = -\ddot{z}(t)/z(t) > 0$ for $t \geqslant t_2$. Thus $\ddot{z}(t) < 0$ for $t > t_2$. Thus $\dot{z}(t_3) < 0$. Considering

$$\left(\frac{\ddot{z}^2}{2z^2} + \dot{z} + \frac{\dot{z}^3}{3z}\right)\bigg|_t = \left(\frac{\ddot{z}^2}{2z^2} + \dot{z} + \frac{\dot{z}^3}{3z}\right)\bigg|_{t_2} + \int_{t_2}^{t} \frac{(\dot{z}(s))^4}{3(z(s))^2}\,\mathrm{d}s$$

the left-hand side $\to -\infty$ as $t \uparrow t_3$ while the right-hand side remains finite (or possibly $\to \infty$) as $t \uparrow t_3$. This proves the lemma. □

Thus for (3.12)− every solution $z(t)$ is eventually of one sign for large t and

$$\frac{\ddot{z}(t)^2}{2z(t)^2} + \frac{\dot{z}(t)^3}{3z(t)} + \dot{z}(t) \uparrow L \qquad \text{as} \quad t \to \infty.$$

If $L = \infty$ then either $z(t) \to 0$ or $z(t) \to \infty$ as $t \to \infty$. If $L < \infty$, the solution is asymptotic to a two-dimensional surface.

For (3.12)+ we begin with the following lemma.

Lemma 2. *For an oscillatory solution of the scalar form of (3.12)+, $x(t) < 0$ for large t.*

Proof. Suppose $z(t)$ is an oscillatory solution. Since $\dot{x}(t) > 0$ for $z(t) > 0$, its graph must have one of two forms shown in figure 1 (recall $x = \ddot{z}/z$).

In the first case $x(t) < 0$ for all t. It will be shown that the second case is impossible. If $x(t_1) > 0$ then $x(t) > 0$ for $t > t_1$ (slightly) and likewise $\dot{z}(t) < 0$ for $t > t_1$ (slightly).

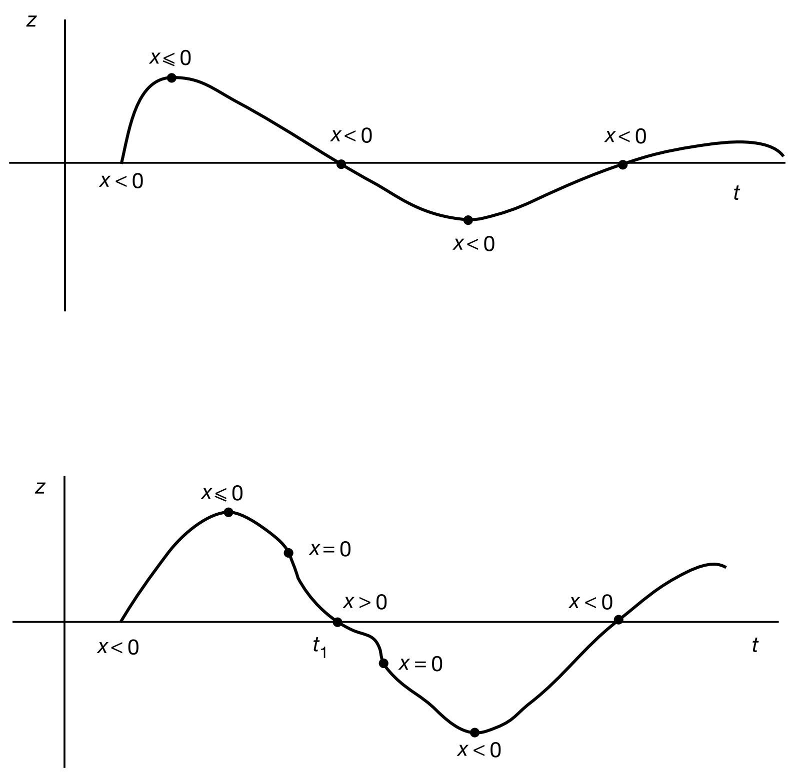

Figure 1.

But then $\dot{z}(t)^2 > z(t)$ for $t \geqslant t_1$ by comparison with the equation in $\dot{u}(t)^2 = u(t)$, $u(t_1) = 0$, whose solution is $u(t) = -\frac{1}{4}(t - t_1)^2$ which is parabolic or flat at $t = t_1$. Thus $x(t)$ remains increasing and positive for $t > t_1$, a contradiction.

To show that (3.12)+ is not chaotic, consider first an oscillatory solution z. Writing $\dot{x} = y^2 + \frac{\dot{y}}{x}$ as $x\dot{x} - \dot{y} = xy^2$ and integrating gives

$$x^2(t) - y(t) = c + \int_{t_0}^{t} x(s)(y(s))^2 \, \mathrm{d}s \downarrow L$$

since $x < 0$. If $L = -\infty$ then $\lim_{t\to\infty} y(t) = \infty$. If $L > -\infty$, then the solution is asymptotic to a two-dimensional surface. For a nonoscillatory solution $z(t) \neq 0$ for $t \geqslant t_0$ consider the earlier expression

$$\frac{\ddot{z}^2}{zz^2} - \frac{\dot{z}^3}{3z} - \dot{z} = c + \int_{t_0}^{t} \frac{(\dot{z}(s))^4}{3(z(s))^2} \, \mathrm{d}s \uparrow L$$

and apply the same argument as for (3.12)−. This completes the proof of the theorem. □

Of all the four-term conservative systems analysed in this paper system (3.3)− is the only one which cannot be completely resolved as nonchaotic. Converting to scalar form we obtain $\dddot{z} = \dot{z}^2 - z^2$.

Lemma 3. *Every nonoscillatory solution of* $\dddot{z} = \dot{z}^2 - z^2$ *is either asymptotic to a two-dimensional surface or converges to* ∞ *in norm* $\|(z, \dot{z}, \ddot{z})\| = \|(z, y, x)\|$ *as* $t \to \infty$.

Proof. Multiplying by $\ddot{z}$ and integrating we obtain

$$\tfrac{1}{2}\ddot{z}^2 - \tfrac{1}{3}\dot{z}^3 + z^2\dot{z} = c + 2\int_{t_0}^{t} (\dot{z}(s))^2 z(s) \, \mathrm{d}s.$$

Suppose $z > 0$ for $t \geqslant t_0$. Then

$$\tfrac{1}{2}\ddot{z}^2 - \tfrac{1}{3}\dot{z}^3 + z^2\dot{z} \uparrow L \qquad \text{as} \quad t \to \infty.$$

If $L < \infty$, the solution is asymptotic to a two-dimensional surface. If $L = \infty$ then $\|(z_1\dot{z}_1\ddot{z})\| \to \infty$ as $t \to \infty$.

If $z < 0$ for $t \geqslant t_0$, then

$$\tfrac{1}{2}\ddot{z}^2 - \tfrac{1}{3}\dot{z}^3 + z^2\dot{z} \downarrow L \qquad \text{as} \quad t \to \infty.$$

If $L > -\infty$, then the solution is asymptotic to a two-dimensional surface. If $L = -\infty$, then clearly $z(t) \to -\infty$ as $t \to \infty$. □

A computer search for oscillatory solutions produces the graphs shown in figures 2 and 3.

These graphs clearly indicate that (3.3)− has a periodic orbit which, moreover, is highly unstable. In fact it appears from computer analysis that there is a unique periodic orbit and that all other solutions diverge to $\pm\infty$. Compare this behaviour with [3].

Another curious fact about system (3.3)− is that it is an example of what is called a reversible dynamical system. Reversing time $t \to -t$ and performing the reflection $z \to -z$ leave the system invariant. Such systems are of much current interest [2] but we do not see how the general theory helps with our specific example.

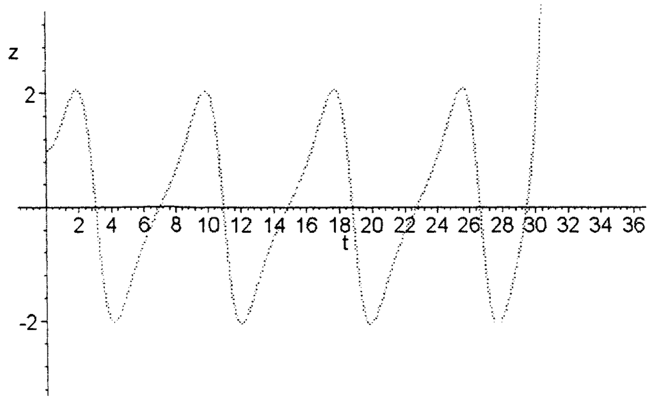

Figure 2. Initial conditions: $\begin{cases} x = 1.330\,754\,572\,23 \\ y = 0 \\ z = 1 \end{cases}$.

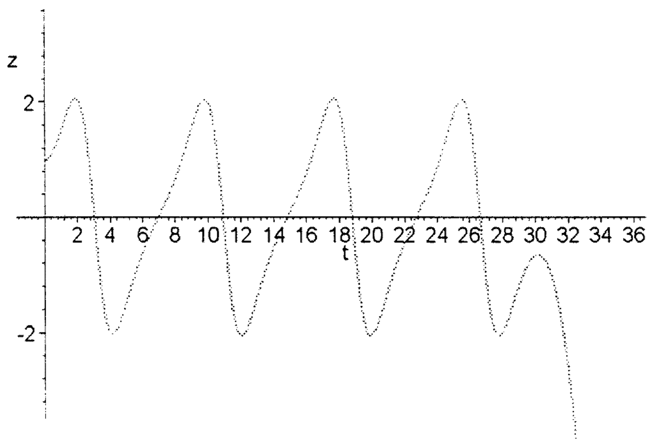

Figure 3. Initial conditions: $\begin{cases} x = 1.330\,754\,572\,22 \\ y = 0 \\ z = 1 \end{cases}$.

4. Four-term systems with three quadratic nonlinearities

The conservative four-term equations with three quadratic nonlinearities are:

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = -2xy \\ \dot{z} = y \end{cases} \tag{4.1}$$

$$\begin{cases} \dot{x} = -2xy + y^2 \\ \dot{y} = y^2 \\ \dot{x} = x \end{cases} \tag{4.2}$$

$$\begin{cases} \dot{x} = -2xy + yz \\ \dot{y} = y^2 \\ \dot{z} = x \end{cases} \tag{4.3}$$

$$\begin{cases} \dot{x} = -2xy + z^2 \\ \dot{y} = y^2 \\ \dot{z} = x \end{cases} \tag{4.4}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = \pm xz \\ \dot{z} = y \end{cases} \tag{4.5}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x^2 \\ \dot{z} = x \end{cases} \tag{4.6}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x^2 \\ \dot{z} = y \end{cases} \tag{4.7}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = z^2 \\ \dot{z} = x \end{cases} \tag{4.8}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = x^2 \\ \dot{z} = x \end{cases} \tag{4.9}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = x^2 \\ \dot{z} = y \end{cases} \tag{4.10}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = xz \\ \dot{z} = y \end{cases} \tag{4.11}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = z^2 \\ \dot{z} = x \end{cases} \tag{4.12}$$

$$\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = z^2 \\ \dot{z} = -2xz \end{cases} \tag{4.13}$$

$$\begin{cases} \dot{x} = xy + y \\ \dot{y} = xz \\ \dot{z} = -yz \end{cases} \tag{4.14}$$

$$\begin{cases} \dot{x} = xy + z \\ \dot{y} = x^2 \\ \dot{z} = -yz \end{cases} \tag{4.15}$$

$$\begin{cases} \dot{x} = xy + z \\ \dot{y} = xz \\ \dot{z} = -yz \end{cases} \tag{4.16}$$

$$\begin{cases} \dot{x} = -2xy + z \\ \dot{y} = y^2 \\ \dot{z} = xy \end{cases} \tag{4.17}$$

$$\begin{cases} \dot{x} = -2xy + z \\ \dot{y} = y^2 \\ \dot{z} = x^2 \end{cases} \tag{4.18}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases} \tag{4.19}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{4.20}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = z^2 \\ \dot{z} = x^2 \end{cases} \tag{4.21}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = z^2 \\ \dot{z} = xy \end{cases} \tag{4.22}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x^2 \\ \dot{z} = xy \end{cases} \tag{4.23}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x^2 \\ \dot{z} = y^2 \end{cases} \tag{4.24}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases} \tag{4.25}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{4.26}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = z^2 \\ \dot{z} = x^2 \end{cases} \tag{4.27}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = z^2 \\ \dot{z} = xy \end{cases} \tag{4.28}$$

Theorem 3. *Systems (4.1)–(4.28) are not chaotic.*

Proof. First of all, notice that systems (4.2), (4.3), (4.14) and (4.17) are essentially two-dimensional and/or linear. Then note that systems (4.4), (4.6)–(4.11)+, (4.12), (4.13), (4.15), and (4.17)–(4.28) all have at least one monotone component which thus has a limit.

The systems with the following scalar forms are easily resolved:

(4.1) $\quad \dfrac{\dddot{z}}{\dot{z}} - \dfrac{1}{2}\left(\dfrac{\ddot{z}}{\dot{z}}\right)^2 + 2\dot{z}z = 0 \quad$ (integrate),

(4.5) $\quad \dddot{z}z \mp \dot{z}^2 z^2 \mp \dot{z}z^3 - \ddot{z}\dot{z} = 0 \quad$ (integrate),

(4.16) $\quad \dddot{y} + 2z^2 y = 0 \quad$ (multiply by y and integrate).

This leaves only system (4.11)− which can be written as

$$\begin{cases} \dot{x} = y^2 - z^2 \\ \ddot{z} = xz. \end{cases}$$

The Ricatti substitution $u = \frac{\dot{z}}{z}$ leads to the system

$$\begin{cases} \dot{u} = x - u^2 \\ \dot{x} = ce^{2\int u(s)\,\mathrm{d}s}(u^2 - 1). \end{cases}$$

Simple graphical analysis of the above (u, x) system shows that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} u(t) = \infty$ since $x(t)$ is increasing (for $u^2 > 1$) more quickly than $u^2(t)$. This concludes the proof of the theorem. □

5. Four-term systems, all nonlinear

The conservative systems with four terms, all nonlinear are:

$$\begin{cases} \dot{x} = x^2 \pm y^2 \\ \dot{y} = z^2 \\ \dot{z} = -2zx \end{cases} \tag{5.1}$$

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = x^2 \\ \dot{z} = -2xz \end{cases} \tag{5.2}$$

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = -2xy \\ \dot{z} = x^2 \end{cases} \tag{5.3}$$

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = -2xy \\ \dot{z} = y^2 \end{cases} \tag{5.4}$$

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = z^2 \\ \dot{z} = -2xz \end{cases} \tag{5.5}$$

$$\begin{cases} \dot{x} = xy + yz \\ \dot{y} = \pm xz \\ \dot{z} = -yz \end{cases} \tag{5.6}$$

$$\begin{cases} \dot{x} = -2xy + yz \\ \dot{y} = y^2 \\ \dot{z} = x^2 \end{cases} \tag{5.7}$$

$$\begin{cases} \dot{x} = xy + z^2 \\ \dot{y} = \pm x^2 \\ \dot{z} = -yz \end{cases} \tag{5.8}$$

$$\begin{cases} \dot{x} = xy + z^2 \\ \dot{y} = xz \\ \dot{z} = -yz \end{cases} \tag{5.9}$$

$$\begin{cases} \dot{x} = -2xy + z^2 \\ \dot{y} = y^2 \\ \dot{z} = x^2 \end{cases} \tag{5.10}$$

$$\begin{cases} \dot{x} = -2xy + z^2 \\ \dot{y} = y^2 \\ \dot{z} = xy \end{cases} \tag{5.11}$$

Theorem 4. *Systems (5.1)–(5.11) are not chaotic.*

Proof. System (5.6) reduces to a two-dimensional system. System (5.9) is equivalent to the scalar equation $\dddot{y} + 3y\ddot{y} = 0$ which is easily seen, by integration, to be asymptotic to a two-dimensional surface.

All of the remaining systems have at least one monotone component and hence are either unbounded or asymptotic to a two-dimensional surface. □

6. Four-term systems with one constant term and one nonlinearity

Because there are so many conservative systems with a constant term we also treat these in separate sections beginning with four-term equations with a constant term and only one nonlinear term. These are:

$$\begin{cases} \dot{x} = 1 + yz \\ \dot{y} = x \\ \dot{z} = y \end{cases} \tag{6.1}$$

$$\begin{cases} \dot{x} = \pm 1 + y^2 \\ \dot{y} = z \\ \dot{z} = x \end{cases} \tag{6.2}$$

$$\begin{cases} \dot x = y^2 + z \\ \dot y = x \\ \dot z = 1 \end{cases} \tag{6.3}$$

$$\begin{cases} \dot x = y^2 \\ \dot y = x + z \\ \dot z = \pm 1 \end{cases} \tag{6.4}$$

$$\begin{cases} \dot x = y^2 \\ \dot y = 1 + z \\ \dot z = \pm x \end{cases} \tag{6.5}$$

$$\begin{cases} \dot x = yz \\ \dot y = 1 + x \\ \dot z = x \end{cases} \tag{6.6}$$

$$\begin{cases} \dot x = y^2 \\ \dot y = z \\ \dot z = 1 + x \end{cases} \tag{6.7}$$

$$\begin{cases} \dot x = yz \\ \dot y = x \\ \dot z = 1 + y \end{cases} \tag{6.8}$$

Theorem 5. *Systems (6.1)–(6.8) are nonchaotic.*

Proof. System (6.1) can be rewritten as the scalar equation $\dddot z = z\dot z + 1$ which integrates to $\ddot z = \frac{1}{2}z^2 + t + c$. Thus $\lim_{t\to\infty} \ddot z(t) = \infty$ and thus $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(y(t) = \lim_{t\to\infty} z(t) = \infty$.

System (6.3) can be written as $\ddot y = y^2 + t + c$ and thus is analogous to (6.1).

All of the other systems have scalar forms, the solutions of which are asymptotic to two-dimensional surfaces:

(6.2)	$\dddot y = y^2 \pm 1$	(multiply by $\dot y$ and integrate),
(6.4)	$\ddot y = y^2 \pm 1$	(multiply by $\dot y$ and integrate),
(6.5)	$\dddot y \mp y^2 = 0$	(multiply by $\dot y$ and integrate),
(6.6)	$\dddot z z - z^2 - z^2\dot z - \ddot z\dot z = 0$	(integrate),
(6.7)	$\dddot y - y^2 = 0$	(multiply by $\dot y$ and integrate),
(6.8)	$\dddot z - z\dot z - z = 0$	(multiply by z and integrate).

□

Remark. *System (6.2)− has been treated in great detail in [3].*

7. Four-term systems with one constant term and two nonlinearities

The four-term systems with a constant term and two nonlinearities are:

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x \\ \dot{z} = 1 \end{cases} \tag{7.1}$$

$$\begin{cases} \dot{x} = y^2 + z^2 \\ \dot{y} = x \\ \dot{z} = 1 \end{cases} \tag{7.2}$$

$$\begin{cases} \dot{x} = 1 + y^2 \\ \dot{y} = xz \\ \dot{z} = y \end{cases} \tag{7.3}$$

$$\begin{cases} \dot{x} = \pm 1 + y^2 \\ \dot{y} = z^2 \\ \dot{z} = x \end{cases} \tag{7.4}$$

$$\begin{cases} \dot{x} = 1 + yz \\ \dot{y} = x^2 \\ \dot{z} = x \end{cases} \tag{7.5}$$

$$\begin{cases} \dot{x} = 1 + yz \\ \dot{y} = x^2 \\ \dot{z} = y \end{cases} \tag{7.6}$$

$$\begin{cases} \dot{x} = 1 + yz \\ \dot{y} = z^2 \\ \dot{z} = x \end{cases} \tag{7.7}$$

$$\begin{cases} \dot{x} = 1 \pm y^2 \\ \dot{y} = x^2 \\ \dot{z} = x \end{cases} \tag{7.8}$$

$$\begin{cases} \dot{x} = 1 + yz \\ \dot{y} = xz \\ \dot{z} = \pm y \end{cases} \tag{7.9}$$

$$\begin{cases} \dot{x} = 1 + z^2 \\ \dot{y} = x^2 \\ \dot{z} = y \end{cases} \tag{7.10}$$

$$\begin{cases} \dot{x} = z^2 \pm 1 \\ \dot{y} = xz \\ \dot{z} = y \end{cases} \tag{7.11}$$

$$\begin{cases} \dot{x} = 1 + y \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases} \tag{7.12}$$

$$\begin{cases} \dot{x} = 1 + y \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{7.13}$$

$$\begin{cases} \dot{x} = 1 + y \\ \dot{y} = z^2 \\ \dot{z} = x^2 \end{cases} \tag{7.14}$$

$$\begin{cases} \dot{x} = 1 + y \\ \dot{y} = z^2 \\ \dot{z} = xy \end{cases} \tag{7.15}$$

$$\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = 1 \\ \dot{z} = -2xz \end{cases} \tag{7.16}$$

$$\begin{cases} \dot{x} = 1 + 2 \\ \dot{y} = x^2 \\ \dot{z} = xy \end{cases} \tag{7.17}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = xz \\ \dot{z} = 1 \end{cases} \tag{7.18}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = 1 \\ \dot{z} = x^2 \end{cases} \tag{7.19}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x^2 \\ \dot{z} = 1 \end{cases} \tag{7.20}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = xz \\ \dot{z} = 1 \end{cases} \tag{7.21}$$

Theorem 6. *Systems (7.1)–(7.21) are nonchaotic.*

Proof. Note that systems (7.1)–(7.8), (7.10), (7.11)+, and (7.12)–(7.21) all have at least one monotone component which hence has a limit as $t \to \infty$. The scalar form of (7.9) is $\dddot{z}z \mp z^2 - z^3\dot{z} - \ddot{z}\dot{z} = 0$ which can be directly integrated.

System (7.11)− can be written as $\dot{x} = (\frac{\dot{y}}{x})^2 - 1$, $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\dot{y}}{x}) = y$ and hence

$$\ddot{x} = 2\left(\frac{\dot{y}}{x}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{x}\right) = \frac{2y\dot{y}}{x}$$

or $\ddot{x}x = 2y\dot{y}$. This integrates to

$$\dot{x}x - y^2 = c + \int_0^t (\dot{x}(s))^2\,\mathrm{d}s \uparrow L$$

as $t \to \infty$. Thus either $\lim_{t\to\infty} x(t) = \infty$ or the solution is asymptotic to a two-dimensional surface. □

8. Four-term systems with a constant term and three nonlinear terms

The four-term systems with a constant term and three nonlinear terms are:

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x^2 \\ \dot{z} = 1 \end{cases} \tag{8.1}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = \pm xz \\ \dot{z} = 1 \end{cases} \tag{8.2}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = x^2 \\ \dot{z} = 1 \end{cases} \tag{8.3}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = xz \\ \dot{z} = 1 \end{cases} \tag{8.4}$$

$$\begin{cases} \dot{x} = xy + 1 \\ \dot{y} = xz \\ \dot{z} = -xz \end{cases} \tag{8.5}$$

$$\begin{cases} \dot{x} = y^2 \pm 1 \\ \dot{y} = z^2 \\ \dot{z} = x^2 \end{cases} \tag{8.6}$$

$$\begin{cases} \dot{x} = y^2 \pm 1 \\ \dot{y} = z^2 \\ \dot{z} = xy \end{cases} \tag{8.7}$$

$$\begin{cases} \dot{x} = yz + 1 \\ \dot{y} = x^2 \\ \dot{z} = xy \end{cases} \tag{8.8}$$

$$\begin{cases} \dot{x} = yz + 1 \\ \dot{y} = x^2 \\ \dot{z} = y^2 \end{cases} \tag{8.9}$$

$$\begin{cases} \dot{x} = \pm 1 + y^2 \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases} \tag{8.10}$$

$$\begin{cases} \dot{x} = \pm 1 + y^2 \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{8.11}$$

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = -2xy \\ \dot{z} = 1 \end{cases} \tag{8.12}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = 1 \\ \dot{z} = x^2 \end{cases} \tag{8.13}$$

Theorem 7. *Systems (8.1)–(8.13) are not chaotic.*

Proof. All of these systems have at least one monotone term, except (8.5), and hence are not chaotic.

For system (8.5) we notice that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(z + xyz) &= \dot{z} + \dot{x}yz + x\dot{y}z + xy\dot{z} \\ &= -yz + xzy^2 + yz + x^2z^2 - xzy^2 = x^2z^2 \end{aligned}$$

Thus

$$z + y\dot{y} = c + \int^t (x(s))^2(z(s))^2\,\mathrm{d}s \uparrow L \leqslant \infty.$$

If $L = \infty$, then, writing $z(t) = ce^{-\int_{t_0}^t y)(s)\,\mathrm{d}s}$, we have

$$ce^{-\int_{t_0}^t y(s)\,\mathrm{d}s} + y(t)\dot{y}(t) \uparrow \infty \qquad \text{as} \quad t \to \infty.$$

Clearly we must have $\int_{t_0}^t y(s)\,\mathrm{d}s \to \pm\infty$ as $t \to \infty$ and thus $\lim_{t\to\infty} z(t) = \pm\infty$. If $L < \infty$, the solution is asymptotic to a two-dimensional surface. $\square$

9. Conclusions

In this paper we have shown that almost all three-dimensional conservative four-term systems of ordinary differential equations with quadratic nonlinearities are nonchaotic. The lone exception, system (3.3)—

$$\begin{cases} \dot{x} = y^2 - z^2 \\ \dot{y} = x \\ \dot{z} = y \end{cases}$$

with scalar form $\dddot{z} = \dot{z}^2 - z^2$ is a reversible dynamical system which appears numerically to have a unique unstable periodic orbit with all other solutions being unbounded. It would be nice to resolve this behaviour analytically.

To go beyond the dissipative and conservative four-term systems appears to be a very formidable problem because of the sheer number of cases, as discussed in [7].

A more reasonable task would be to perform an exhaustive analysis of three-dimensional systems with five terms and only one (quadratic) nonlinearity, restricted to the dissipative and conservative cases. Sprott [5, 6] has given examples of chaotic systems in both cases. On the other hand the methods of this paper can be used to show that many five-term equations with only one nonlinearity are nonchaotic. The question is: are Sprott's examples the only chaotic ones or are there others as well?

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