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ON THE LOCATION AND NATURE OF DERIVATIVE BLOWUPS OF SOLUTIONS TO CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we consider certain nonlinear differential equations with fixed singularities in the dependent variables which cause derivative blowup in the solution to the corresponding initial value problem. We are interested in the location and nature of derivative blowups. Some theorems and numerical examples are given.

1. Introduction

The nonlinear differential equation

$$y'' = p(x)g(y) \tag{1.1}$$

has been considered by many authors. The special case with $g(y) = y^n$, n real, is called the *Emden-Fowler case* and many papers have been written concerning this case. However, only a few authors were concerned with locations of movable singularities to solutions of the corresponding initial value problem (IVP):

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$$y'' = p(x)g(y), \quad y(x_0) = y_0, \quad y'(x_0) = y_0'. \tag{1.2}$$

See, for example, [1], [3], [5], [6] and [8]. These authors assume that $g(y)$ has no singularity in variable y (such as the Emden-Fowler case). If, however, $g(y)$ is allowed to have a fixed singularity in y, then derivative blowup is possible without blowup in the actual solution $(y(x))$ itself. In [7], Kawarada considered such a case, although the nonlinear differential equation is of first order. It has the form

$$y' = \frac{1}{A - y}, \quad y(0) = y_0, \quad y_0 < A. \tag{1.3}$$

The behavior of this IVP, as pointed out in [7], sheds light on the behavior of the solution to certain PDEs of importance. Clearly, (1.3) can be written $y'' = \frac{1}{(A-y)^2}$, as

$$y'' = \frac{1}{(A-y)^2}, \quad y(0) = y_0, \quad y'(0) = y_0', \tag{1.4}$$

where $y_0' = (A - y_0)^{-1}$. Clearly, $y''(x) \to \infty$ as $x \to c^-$ for some $c > x_0$ if $y_0' > 0$. We shall refer to this occurrence as 'derivative blowup' in the sequel. We wish to obtain information on the location of c, a point at which a movable singularity occurs to the solution. Many papers have been written on existence of blowup in PDEs, but no references will be given here. Our primary interest is on the nature and location of the blowup.

2. A Second Order Autonomous Case

In this section, we shall obtain upper and lower bounds on c, the location of the movable singularity of derivative blowup type. Consider the IVP:

$$y'' = p(x)[g(y) \cdot (A - y)^{\delta}], \quad y(x_0) = y_0, \quad y'(x_0) = y_0', \tag{2.1}$$

with $\delta < 0$, A real, $y_0 \langle A, y_0' \rangle 0$, $g(y)$ is continuous in y, $g(y) > 0$ on $[y_0, \infty)$, and $p(x)$ is continuous and positive on $[x_0, \infty)$. Assume that there exists a $c > x_0$ with $\lim_{x \to c^-} y''(x) = \infty$. (This would be the case, for example, if there exists $P_0 > 0$ with $p(x) \geq P_0$, $x \geq x_0$.) We have the following main result:

Theorem 1. *Consider* IVP (2.1) *under the stated conditions above*. *Let*

$$\widetilde{Q}_L(x) = \inf_{x_0 < t < x} p(t) \cdot \inf_{y_0 < w < A} g(w). \tag{2.2}$$

Suppose $\delta < -1$. *Then an upper bound for c is implicitly given by*

$$\int_{y_0}^{A} \sqrt{\frac{\delta + 1}{(A - y_0)^{\delta+1} - (A - y)^{\delta+1}}} dy \geq \int_{x_0}^{c} \sqrt{2\widetilde{Q}_L(x)}\, dx. \tag{2.3}$$

If $\delta = -1$, *then an upper bound for c is implicitly given by*

$$\int_{y_0}^{A} \frac{dy}{\sqrt{Ln(A - y_0) - Ln(A - y)}} \geq \int_{x_0}^{c} \sqrt{2\widetilde{Q}_L(x)}\, dx. \tag{2.4}$$

Proof. We shall prove (2.3). The proof of (2.4) is similar and is omitted. Then multiplication of (2.1) by y' gives

$$y''(t)\, y'(t) = p(t)\, g(y(t))\, (A - y(t))^{\delta}\, y'(t).$$

Integrating with respect to *t* from $t = x_0$ to $t = x$ gives

$$\frac{1}{2}(y'(x))^2 - \frac{1}{2}(y_0')^2 = \int_{x_0}^{x} p(t)\, g(y(t))\, (A - y(t))^{\delta}\, y'(t)\, dt$$

$$\geq \widetilde{Q}_L(x) \cdot \int_{x_0}^{x} (A - y(t))^{\delta}\, y'(t)\, dt,$$

using (2.2) given

$$(y'(x))^2 \geq (y_0')^2 + 2\widetilde{Q}_L(x) \int_{x_0}^{x} (A - y(t))^{\delta}\, y'(t)\, dt$$

$$\geq 2\widetilde{Q}_L(x) \int_{x_0}^{x} (A - y(t))^{\delta}\, y'(t)\, dt$$

$$= 2\widetilde{Q}_L(x) \cdot \frac{1}{\delta + 1}[(A - y_0)^{\delta+1} - (A - y(x))^{\delta}],$$

from which the result follows, upon letting $x \to c^-$ and integrating once more with respect to *x*. This completes the proof of (2.3).

Remark 1. Formulas for lower bounds for c can be found in [6]. First, we need the following inequality. See, [9, Chapter X], for example.

Gruss Inequality. Let $f_1(x)$ and $f_2(x)$ be integrable on $[a, b]$.

(a) If $f_1(x)$ and $f_2(x)$ are both nondecreasing or both nonincreasing on $[a, b]$, then

$$\int_a^b f_1(x) f_2(x) dx \geq (b-a)^{-1} \int_a^b f_1(x) ds \times \int_a^b f_2(x) dx.$$

(b) If $f_1(x)$ is nondecreasing and $f_2(x)$ is nonincreasing on $[a, b]$, then

$$\int_a^b f_1(x) f_2(x) dx \leq (b-a)^{-1} \int_a^b f_1(x) ds \times \int_a^b f_2(x) dx.$$

In the case, where $g(y)$ is nondecreasing in y and $p(x)$ is nondecreasing in x, we present next an alternative upper bound for c, for $\delta < -1$.

Theorem 2. *If, in addition to the conditions of Theorem* 1, *we also have that* $p(x)$ *is nondecreasing on* $[x_0, \infty)$ *and* $g(y)$ *is nondecreasing on* $[y_0, A]$, *then an upper bound for c is implicitly given by*

$$\int_{y_0}^{A} \sqrt{\frac{\delta + 1}{(A - y_0)^{\delta+1} - (A - y)^{\delta+1}}} dy \geq \int_{x_0}^{c} \widetilde{Q}(w) dw, \tag{2.5}$$

where

$$\widetilde{Q}(w) = \sqrt{\frac{2g(y_0)\int_{x_0}^{w} p(u) du}{w - x_0}}.$$

Proof. Proceeding as in the proof of (2.3),

$$y''(u) y'(u) = p(u) g(y(u)) y'(u)(A - y(u))^{\delta}. \tag{2.6}$$

Integration gives, using the Gruss inequality:

$$\frac{1}{2}[(y'(t))^2 - (y_0')^2] \geq \frac{\int_{x_0}^{t} p(u) du}{t - x_0} \cdot \int_{x_0}^{t} g(y(u))(A - y(u))^{\delta} y'(u) du. \tag{2.7}$$

Since $g(y(t)) \geq g(y_0)$, we obtain

$$y'(t) \geq \widetilde{Q}(t)dt \cdot \sqrt{\frac{1}{\delta+1}[(A-y_0)^{\delta+1} - (A-y(t))^{\delta+1}]}. \tag{2.8}$$

Integration of (2.8) produces (2.5). This completes the proof. □

Finally, we present an 'obvious' upper bound for c in Theorem 3 below. We shall also consider this upper bound in some numerical examples later.

Theorem 3. *Under the conditions of Theorem* 1, *an upper-bound for c is*

$$c_3 = x_0 + \frac{A - y_0}{y_0'}. \tag{2.9}$$

Proof. Since $y'' \geq 0$ on $[x_0, c)$, we have $y'(x)$ increasing on $[x_0, c)$ and $y_0 + y' + 0(x - x_0) \leq A$, which gives $x \leq c_3$, as desired. □

Numerical examples demonstrate that (2.5) usually provides a better (smaller) upper bounds than (2.3) and (2.9), if $\delta < -1$. However, (2.3) is more generally applicable. One numerical example is presented below.

Example 1. Consider the IVP

$$y'' = (1 + x^2)(1 - y)^{-2}, \quad y_0 = 0.1, \quad y_0' = 2.0.$$

Then (2.3) gives an upper bound for c of 0.9483, whereas (2.5) gives an upper bound of 0.8234. Runge-Kutta fourth order method found $c \approx 0.33$. Expression (2.9) gives an upper bound of 0.4500 and is the best. However, for small values of y_0', (2.5) and (2.3) usually provide better upper bounds for c, if $p(x)$ is nondecreasing.

3. Competing Blowup Causes

Consider the IVP

$$y'' = p(x)(A - y)^{\delta_1}(B - y')^{\delta_2}, \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \tag{3.1}$$

where $-\infty < y_0 < A$, $0 < y_0' < B$. Suppose (3.1) has a solution with derivative blowup at $x = c$, i.e., $\lim_{x \to c^-} y''(x) = \infty$. It is clear that if $p(x) \geq P_0 > 0$ on $[x_0, \infty)$, then there will be a derivative blowup, since $y'(x) \geq y_0' > 0$ and $y'' > 0$.

In Section 2, we assumed $\delta_2 = 0$. Here, we require $\delta_1 < 0$ and $\delta_2 < 0$. Allowing δ_2 to be nonzero and negative allows for a sort of 'competition' between the $(A - y)^{\delta_1}$ term and the $(B - y')^{\delta_2}$ term to be the 'cause' of the blowup of y'' in (3.1). No such 'competition' occurs in the autonomous case considered earlier in Section 2. Does $y'' \to \infty$ as $x \to c^-$ because $y \to A$ as $x \to c^-$ or $y' \to B$ as $x \to c^-$? In addition, what can be said about the locations of c? We aim to answer, at least partially, these questions next.

First, we consider the case $p(x) \equiv 1$, to gain some insight into the problem. This case also allows us to solve for $y(x)$ for some choices of δ_1 and δ_2. We also consider a few special cases of an IVP containing powers of y and or y', i.e., the IVP

$$y'' = y^m (A - y)^{-\alpha} (y')^{\beta} (B - y')^{-n}, \tag{3.2}$$

$y(x_0) = y_0$, $y'(x_0) = y_1$, $\alpha > 0$, $m > 0$, $n > 0$, $\beta > 0$, $0 < y_0 < A$, $0 < y_1 < B$, m, n integers.

Theorem 4. *Consider* IVP (3.2) *with* $y_0 = y_1 = 0$, $x_0 = 0$. *Suppose* $0 < \alpha < 1$. *Then*

(a) *blowup occurs in* y'' *at* $x = c$ *because* $y(x) \to A$ *as* $x \to c^-$, *if*

$$\frac{m!\, A^{m+1-\alpha}}{(1-\alpha)(2-\alpha)\cdots(m+1-\alpha)} < \frac{n!\, B^{n+2-\beta}}{(2-\beta)(3-\beta)\cdots(n+2-\beta)},$$

(b) *blowup occurs in* y'' *at* $x = c$ *because* $y'(x) \to B$ *as* $x \to c^-$, *if*

$$\frac{n!\, B^{n+2-\beta}}{(2-\beta)(3-\beta)\cdots(n+2-\beta)} < \frac{M!\, A^{m+1-\alpha}}{(1-\alpha)(2-\alpha)\cdots(m+1-\alpha)}.$$

Proof. We prove the result for $m = 3$ and $n = 2$ to ease the laborious algebra and the notational complexity, as well as for simplicity and without loss of generality,

$$y'' = \frac{y^3}{(A - y)^{\alpha}} \frac{(y')^{\beta}}{(B - y')^2}$$

gives

$$\frac{y''(B-y')^2}{(y')^{\beta-1}} = \frac{y^3}{(A-y)^{\alpha}}\,y'. \tag{3.3}$$

Integration of the L.H.S. of (3.3) gives

$$\int \frac{(B-y')^2}{(y')^{\beta-1}}\,y'' = \int \frac{\beta^2 - 2\beta y' + (y')^2}{(y')^{\beta-1}}\,y''$$

$$= \frac{B^2(y')^{2-\beta}}{2-\beta} - \frac{2B(y')^{3-\beta}}{3-\beta} + \frac{(y')^{4-\beta}}{4-\beta}. \tag{3.4}$$

Integration of the R.H.S. of (3.4) gives, upon making the substitution $u = A - y$,

$$\int \frac{y^3}{(A-y)^{\alpha}}\,y' = -\int \frac{(A-u)^3 u'}{u^{\alpha}}$$

$$= -\int (A^3u^{-\alpha} - 3A^2u^{1-\alpha} + 3Au^{2-\alpha} - u^{3-\alpha})u'. \tag{3.5}$$

Equating (3.4) and (3.5) up to a constant of integration, c, we obtain

$$\frac{B^2(y')^{2-\beta}}{2-\beta} - \frac{2B(y')^{3-\beta}}{3-\beta} + \frac{(y')^{4-\beta}}{4-\beta}$$

$$+ \frac{A^3(A-y)^{1-\alpha}}{1-\alpha} - \frac{3A^2(A-y)^{2-\alpha}}{2-\alpha}$$

$$+ \frac{3A(A-y)^{3-\alpha}}{3-\alpha} - \frac{(A-y)^{4-\alpha}}{4-\alpha}$$

$$= c. \tag{3.6}$$

Using $y(0) = y'(0) = 0,$ we obtain

$$c = \frac{6A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}.$$

Thus

$$\frac{B^2(y')^{2-\beta}}{2-\beta} - \frac{2B(y')^{3-\beta}}{3-\beta} + \frac{(y')^{4-\beta}}{4-\beta} + \frac{A^3(A-y')^{1-\alpha}}{1-\alpha}$$

$$-\frac{3A^2(A-y)^{2-\alpha}}{2-\alpha}+\frac{3A(A-y)^{3-\alpha}}{3-\alpha}-\frac{(A-y)^{4-\alpha}}{4-\alpha}$$

$$=\frac{A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}. \tag{3.7}$$

Suppose $y'(x)\to B$ as $x\to c^-$ with $y(c^-)<A.$ Then

$$\frac{B^{4-\beta}}{2-\beta}-\frac{2B^{4-\beta}}{3-\beta}+\frac{B^{4-\beta}}{4-\beta}+\frac{A^3(A-y)^{1-\alpha}}{1-\alpha}$$

$$-\frac{3A^2(A-y)^{2-\alpha}}{2-\alpha}+\frac{3A(A-y)^{3-\alpha}}{3-\alpha}-\frac{(A-y)^{4-\alpha}}{4-\alpha}$$

$$=\frac{6A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}.$$

But then

$$0<\frac{A^3(A-y)^{1-\alpha}}{1-\alpha}-\frac{3A^2(A-y)^{2-\alpha}}{2-\alpha}+\frac{3A(A-y)^{3-\alpha}}{3-\alpha}-\frac{(A-y)^{4-\alpha}}{4-\alpha}$$

$$=\frac{6A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}-\frac{2B^{4-\beta}}{(2-b\eta)(3-\beta)(4-\beta)}.$$

Thus,

$$\frac{2B^{4-\beta}}{(2-\beta)(3-\beta)(4-\beta)}\le\frac{6A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}.$$

So part (a) is proven.

Now suppose $y(x)\to A$ as $x\to c^-$ with $y'(c^-)<B.$ Then

$$\frac{B^{4-\beta}}{2-\beta}-\frac{2B^{4-\beta}}{3-\beta}+\frac{B^{4-\beta}}{4-\beta}$$

$$>\frac{B^2(y')^{2-\beta}}{2-\beta}-\frac{2B(y')^{3-\beta}}{3-\beta}+\frac{(y')^{4-\beta}}{4-\beta}$$

$$=\frac{6A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}.$$

Thus

$$\frac{6A^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)} \leq \frac{2B^{4-\beta}}{(2-\beta)(3-\beta)(4-\beta)}.$$

So (b) is proven. □

Theorem 5. *Consider the* IVP

$$y'' = \frac{y^m}{(A-y)^\alpha} \frac{(y')^n}{(B-y')^\beta}, \tag{3.8}$$

where $A, B > 0,\ \alpha \geq 1,\ \beta < 1,$ *m and n are positive integers*. *Then* $\lim_{x\to c^-} y''(x) = \infty$ *occurs because* $\lim_{x\to c^-} y'(x) = B.$

Proof. Again, for simplicity but without loss of generality, let $m = \beta = 2.$ Then

$$y'' = \frac{y^2}{(A-y)^\alpha} \frac{(y')^n}{(B-y')^\beta},$$

which gives

$$\frac{(B-y')^2}{(y')^{n-1}} y'' = \frac{y^2 y^1}{(A-y)^\alpha}. \tag{3.9}$$

Integrating both sides of (3.9), we obtain

$$\int \frac{(B-y')^2 y''}{(y')^{n-1}} = \frac{B^2}{2-n}(y')^{2-n} - \frac{2B}{3-n}(y')^{3-n} + \frac{1}{4-n}(y')^{4-n}. \tag{3.10}$$

Also,

$$\int \frac{y^2 y^1}{(A-y)^\alpha} = -\frac{A^2}{1-\alpha}(A-y)^{1-\alpha} + \frac{2A}{2-\alpha}(A-y)^{2-\alpha} - \frac{(A-y)^{3-\alpha}}{3-\alpha}. \tag{3.11}$$

Equating (3.10) and (3.11) up to an arbitrary constant, c, of integration, we have

$$\frac{B^2}{2-n}(y')^{2-n}$$

$$= \frac{2B}{3-n}(y')^{3-n} + \frac{(y')^{4-n}}{4-n}$$

$$= -\frac{A^2}{1-\alpha}(A-y)^{1-\alpha} + \frac{2A}{2-\alpha}(A-y)^{2-\alpha} - \frac{(A-y)^{3-\alpha}}{3-\alpha} + C. \tag{3.12}$$

Since $\alpha > 1$, $\lim_{x\to c^-} y(x) = A$ is not compatible with (3.12), a contradiction. Thus, $\lim_{x\to c^-} y'(x) = B$. This completes the proof. □

Theorem 6. *Consider the* IVP

$$y'' = \frac{1}{(A-y)^{\alpha}} \cdot \frac{1}{(B-y')^{\beta}}, \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \tag{3.13}$$

$A > 0$, $B > 0$, $\alpha \geq 1$, $\beta > 0$, $y_1 > 0$. *Then derivative blowup occurs because* $\lim_{x\to c^-} y'(x) = B$.

Proof. Write

$$y''(B-y')^{\beta} y' = \frac{y'}{(A-y)^{\alpha}}.$$

Integration of both sides gives, eventually,

$$-\frac{B}{\beta+1}(B-y')^{\beta+1} + \frac{(B-y')^{\beta+2}}{\beta+2} = -\frac{(A-y)^{1-\alpha}}{1-\alpha} + C. \tag{3.14}$$

Thus, since $\alpha > 1$, $\lim_{x\to c^-} y(x) = A$ is a contradiction to (3.14). □

Theorem 7. *Consider the* IVP

$$y'' = \frac{1}{(A-y)^{\alpha}} \cdot \frac{1}{(B-y')^{\beta}},$$

where A, B, α *and* β *are real with* $\beta > 0$, $y(0) = 0$, $y'(0) = 0$. *Then blowup occurs at* $x = c$, *where* $\lim_{x\to c^-} y'(x) = B$ *provided*

$$A^{1-\alpha} < \frac{1-\alpha}{(\beta+1)(\beta+2)} B^{\beta+2}. \tag{3.15}$$

Proof. Write

$$y''(B-y')^{\beta} y' = \frac{y'}{(A-y)^{\alpha}}.$$

But

$$\int (B - y')^{\beta} y' y'' = -\frac{B}{\beta + 1}(B - y')^{\beta+1} + \frac{1}{\beta + 2}(B - y')^{\beta+2}.$$

Then

$$\frac{B}{\beta + 1}(B - y')^{\beta+1} + \frac{1}{\beta + 2}(B - y')^{\beta+2} + \frac{1}{1 - \alpha}(A - y)^{1-\alpha} = C,$$

where

$$C = -\frac{B}{\beta + 1} B^{\beta+1} + \frac{1}{\beta + 2} B^{\beta+2} + \frac{1}{1 - \alpha} A^{1-\alpha}.$$

Thus,

$$-\frac{B}{\beta + 1}(B - y')^{\beta+1} + \frac{1}{\beta + 2}(B - y')^{\beta+2} + \frac{1}{1 - \alpha}(A - y)^{1-\alpha}$$

$$= \frac{1}{1 - \alpha} A^{1-\alpha} - \left(\frac{1}{\beta + 1} - \frac{1}{\beta + 2}\right) B^{\beta+2}.$$

Now suppose $y' = B$ and $y < A$. Then

$$\frac{1}{1 - \alpha}(A - y)^{1-\alpha} = \frac{1}{1 - \alpha} A^{1-\alpha} - \frac{B^{\beta+2}}{(\beta + 1)(\beta + 2)}$$

which eventually gives

$$y = A - \left[A^{1-\alpha} - \frac{(1 - \alpha) B^{\beta+2}}{(\beta + 1)(\beta + 2)}\right]^{\frac{1}{1-\alpha}}.$$

Thus

$$\frac{(1 - \alpha) B^{\beta+2}}{(\beta + 1)(\beta + 2)} < A^{1-\alpha}.$$

Now suppose $y = A$ and $0 < y' < B$. Then

$$0 > -\frac{B}{\beta + 1}(B - y')^{\beta+1} + \frac{1}{\beta + 2} B(B - y')^{\beta+2}$$

$$= \frac{1}{1 - \alpha} A^{1-\alpha} - \frac{B^{\beta+2}}{(\beta + 1)(\beta + 2)}$$

giving $A^{1-\alpha} < \dfrac{(1-\alpha)B^{\beta+2}}{(\beta+1)(\beta+2)}$, as desired. This completes the proof.

We now consider a case in which *c*, the blowup point, can be explicitly found as a definite integral.

Theorem 8. *Consider the* IVP

$$y'' = \frac{1}{(A-y)^{\alpha}} \frac{y'}{(B-y')^{\beta}}, \tag{3.16}$$

$y(0) = 0,\ \ y'(0) = 0,\ \ A, B, \alpha, \beta > 0$ *and* $0 < \alpha < 1$. *Then*

(a) *blowup occurs because* $y(x) \to A$ *as* $x \to c^-$, *if*

$$(1-\alpha)B^{\beta+1} > (\beta+1)A^{1-\alpha} \tag{3.17}$$

and

(b) *blowup occurs because* $y'(x) \to B$ *as* $x \to c^-$, *if*

$$(1-\alpha)B^{\beta+1} > (\beta+1)A^{1-\alpha}. \tag{3.18}$$

Proof. Write

$$y''(B-y')^{\beta} = \frac{y'}{(A-y)^{\alpha}}.$$

Integration gives

$$\int y''(B-y')^{\beta} = -\frac{(B-y')^{\beta+1}}{\beta+1}.$$

Thus

$$-\frac{(B-y')^{\beta+1}}{\beta+1} = -\frac{(A-y)^{1-\alpha}}{1-\alpha} + C,$$

where

$$C = \frac{A^{1-\alpha}}{1-\alpha} - \frac{B^{\beta+1}}{\beta+1}.$$

Now suppose $y(x) \to A$ as $x \to c^-$. Then

$$\frac{(B - y')^{\beta+1}}{\beta + 1} = \frac{B^{\beta+1}}{\beta + 1} - \frac{A^{1-\alpha}}{1 - \alpha}$$

or

$$y' = B - \left(B^{\beta+1} - \frac{\beta + 1}{1 - \alpha} A^{1-\alpha} \right)^{\frac{1}{\beta+1}}.$$

So $(1-\alpha) B^{\beta+1} > (\beta+1) A^{1-\alpha}$ implies $y' < B$. Conversely, suppose $\lim_{x \to c^-} y'(x) = B$. Then

$$\frac{(A - y)^{1-\alpha}}{1 - \alpha} = \frac{A^{1-\alpha}}{1 - \alpha} - \frac{B^{\beta+1}}{\beta + 1}$$

or

$$y = A - \left(A^{1-\alpha} - \frac{1-\alpha}{\beta+1} B^{\beta+1} \right)^{\frac{1}{1-\alpha}}.$$

So $(1 - \alpha) B^{\beta+1} < (\beta + 1) A^{1-\alpha}$ implies $\lim_{x \to c^-} y(x) < A$. □

Corollary 9. *Consider the* IVP *of Theorem* 8. *If the* IVP *blows up because* $\lim_{x \to c^-} y(x) = A$, *then the blowup point is explicitly given by*

$$C = \int_0^A \frac{dy}{B - \left[\frac{\beta + 1}{1 - \alpha} (A - y)^{1-\alpha} + B^{\beta+1} - \frac{\beta + 1}{1 - \alpha} A^{1-\alpha} \right]^{\frac{1}{(\beta+1)}}}. \tag{3.19}$$

Proof. From the proof of Theorem 8,

$$\frac{(B - y')^{\beta+1}}{\beta + 1} = \frac{(A - y)^{1-\alpha}}{1 - \alpha} + \frac{B^{\beta+1}}{\beta + 1} - \frac{A^{1-\alpha}}{1 - \alpha}.$$

Then

$$y' = B - \left[\frac{\beta + 1}{1 - \alpha} (A - y)^{1-\alpha} + B^{\beta+1} - \frac{\beta + 1}{1 - \alpha} A^{1-\alpha} \right]^{\frac{1}{\beta+1}}.$$

So

$$C = \int_0^c dx = \int_0^c \frac{y' dx}{B - \left[\frac{\beta - 1}{1 - \alpha}(A - y)^{1-\alpha} + B^{\beta+1} - \frac{\beta + 1}{1 - \alpha} A^{1-\alpha}\right]^{\frac{1}{\beta+1}}},$$

from which (3.19) follows. This completes the proof. □

Corollary 10. *If blowup occurs in* (3.16) *because* $\lim_{x \to c^-} y'(x) = B,$ *then*

$$C = \int_0^V \frac{dy}{B - \left[\frac{\beta - 1}{1 - \alpha}(A - y)^{1-\alpha} + B^{\beta+1} - \frac{\beta + 1}{1 - \alpha} A^{1-\alpha}\right]^{\frac{1}{(\beta+1)}}},$$

where

$$V = A - \left[A^{1-\alpha} - \frac{1 - \alpha}{\beta + 1} B^{\beta+1}\right]^{\frac{1}{1-\alpha}}.$$

Proof. From the previous discussion, we see that when $y' = B,$ then

$$y = A - \left[A^{1-\alpha} - \frac{1 - \alpha}{\beta + 1} B^{\beta+1}\right]^{\frac{1}{1-\alpha}},$$

since

$$C = \int_0^c dx = \int_0^c \frac{y'(x) dx}{B - \left[\frac{\beta + 1}{1 - \alpha}(A - y)^{1-\alpha} B^{\beta+1} - \frac{\beta + 1}{1 - \alpha} AS^{1-\alpha}\right]^{\frac{1}{\beta+1}}}$$

which proves the desired result. □

We shall now consider nonconstant choices for $p(x)$ in (3.1) and a more general form for the IVP than was considered in earlier sections.

Consider the IVP:

$$y'' = p(x) h(y) g(y') (A - y)^{\delta_1} (B - y')^{\delta_2},$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad -1 < \delta_1 < 0, \quad \delta_2 < 0,$$
$$y_0 < A, \quad y_1 < B, \quad B > 0. \tag{3.20}$$

Assume that p, g and h are positive functions of with p, g and h nondecreasing on

$[x_0, c]$ and the p, g and h are continuously differentiable on $[x_0, c)$. Clearly, blowup in either y or y' will produce blowup in y''. In order to prove theorems concerning the nature of the blowup, we will need an upper bound on c, where either

$$\lim_{x \to c^-} y(x) = A \quad \text{or} \quad \lim_{x \to c^-} y'(x) = B.$$

The case where $\delta_1 = \delta_2 = 0$ has been considered in [10], but apparently no extensions/generalizations have been considered.

Lemma 11. *In* (3.20), *an upper bound on C is* $C^* = \max(C_1, C_2)$, *where*

$$C_1 = x_0 + \frac{A - y_0}{y_1},$$

$$C_2 = x_0 + \frac{B - y_1}{y_2},$$

and

$$y_2 = p(x_0) h(y_0) g(y_1) (A - y_0)^{\delta_1} (B - y_1)^{\delta_2}.$$

Proof. Clearly, for $x_0 \le x \le c$,

$$A \ge y(x) \ge y_0 + y_1(x - x_0).$$

For $x = c$, solving for c, we obtain $C = C_1 = x_0 + \dfrac{A - y_0}{y_1}$. Also, $B \ge y'(x) \ge y_1 + y_2(x - x_0)$, where $y_2 = y''(x_0)$, since $y^{(3)}(x) \ge 0$, y'' being a product of five nonnegative nondecreasing differentiable functions of x. □

Lemma 12. *Consider* IVP (3.20) *and the conditions given there*. *Suppose*

$$p(x_0) \int_{y_0}^{A} h(w) (A - w)^{\delta_1} dw < \int_{y_1}^{B} \frac{z dz}{g(z) (B - z)^{\delta_2}} \tag{3.21}$$

holds. *Then an upper bound for c denoted by* C^{**} *is implicitly given by*

$$\int_{y_1}^{B} \frac{z dz}{g(z) (B - z)^{\delta_2}} \ge \left(\int_{y_0}^{A} h(w) (A - w)^{\delta_1} dw \right) \cdot \left(\frac{\int_{x_0}^{c} p(t) dt}{c - x_0} \right), \tag{3.22}$$

provided (3.22) *has at least one solution*.

Proof. Suppose first that $\lim_{x\to c^-} y(x) = A.$ Then (3.20) gives

$$\int_{x_0}^{x} \frac{y'(t)y''(t)dt}{g(y'(t))(B-y'(t))^{\delta_2}} = \int_{x_0}^{x} p(t)h(y(t))\cdot(A-y(t))^{\delta_1} y'(t)dt.$$

Letting $w = y(x)$ and $z = y'(x)$ and using the Gruss inequality,

$$\int_{y_1}^{y'(z)} \frac{zdz}{g(z)(B-z)^{\delta_2}} \geq \left(\int_{y_0}^{y(x)} h(w)(A-w)^{\delta_1}dw\right)\cdot\left(\frac{\int_{x_0}^{x} p(t)dt}{x-x_0}\right). \tag{3.23}$$

Now let $x \to c^-$. Then

$$\int_{y_1}^{B} \frac{zdz}{g(z)(B-z)^{\delta_2}} \geq \int_{y_1}^{y'(x)} \frac{zdz}{g(z)(B-z)^{\delta_2}}$$

$$\geq \left(\int_{y_1}^{A} h(w)(A-w_1)^{\delta_1}dw\right)\cdot\left(\frac{\int_{x_0}^{c} p(t)dt}{c-x_0}\right). \tag{3.24}$$

If $\lim_{x\to c^-} y'(x) = B$ is the cause of the blowup instead, then (3.23) gives instead as $x \to c^-$:

$$\int_{y_1}^{B} \frac{zdz}{g(z)(B-z)^{\delta_2}} \geq \left(\int_{y_0}^{y(c)} h(w)(A-w)^{\delta_1}dw\right)\cdot\left(\frac{\int_{x_0}^{c} p(t)dt}{c-x_0}\right). \tag{3.25}$$

Thus, an upper bound for c is the larger of the two solutions (largest values of c) satisfying both (3.22) and (3.25). Clearly, any solution of (3.22) also satisfies (3.25), since $y(c) \leq A.$ Thus, (3.22) implicitly defines an upper bound for c, regardless of the nature of the blowup, provided (3.22) has at least one solution. But this is the case by assumption (3.21), since $p(x_0) \leq \frac{\int_{x_0}^{c} p(t)dt}{c-x_0}$. This completes the proof. □

Theorem 13. *For* IVP (3.20),

(a) *if* $p(x_0)\int_{y_0}^{A} h(w)(A-w)^{\delta_1}\,dw > \int_{y_1}^{B} \frac{zdz}{g(z)\cdot(B-z)^{\delta_2}}$, *then blowup occurs because* $\lim_{x\to c^-} y'(x) = B$.

(b) *if* $p(c_u)\int_{y_0}^{A} h(w)(A-w)^{\delta_2}\,dw < \int_{y_1}^{B} \frac{zdz}{g(z)\cdot(B-z)^{\delta_2}}$, *where* c_u *is any upper bound for* c, *then blowup occurs because* $\lim_{x\to c^-} y(x) = A$. (*We may use either* $c_u = c^*$ *from Lemma* 11 *or* $c_u = C^{**}$ *from Lemma* 12.)

Proof. To prove (a), suppose $\lim_{x\to c^-} y'(x) \neq B$. From (3.20), we have

$$\int_{x_0}^{x} \frac{y'(t)\,y''(t)}{g(y'(t))\,(B-y'(t))^{\delta_2}}\,dt = \int_{x_0}^{x} p(t)\,h(y(t))\,(A-y(t))^{\delta_1}\,y'(t)\,dt.$$

Letting $z = y'$, $w = y$, we get, using the Gruss inequality

$$\int_{y_1}^{y'(x)} \frac{zdz}{g(z)(B-z)^{\delta_2}} \geq \left(\int_{y_0}^{y(x)} h(w)(A-w)^{\delta_1}\,dw\right)\cdot\left(\frac{\int_{x_0}^{x} p(t)\,dt}{x-x_0}\right)$$

which gives

$$\int_{y_1}^{B} \frac{zdz}{g(z)(B-z)^{\delta_2}} \geq \left(\int_{y_0}^{y(x)} h(w)\cdot(A-w)^{\delta_1}\,dw\right)\cdot\left(\frac{\int_{x_0}^{x} p(t)\,dt}{x-x_0}\right).$$

Since $\lim_{x\to c^-} y(x) = A$, we have, by continuity, for $x_0 \leq x \leq c$:

$$\int_{y_1}^{B} \frac{zdz}{g(z)(B-z)^{\delta_2}} \geq \left(\int_{y_0}^{A} h(w)(A-w)^{\delta_1}\,dw\right)\cdot\left(\frac{\int_{x_0}^{x} p(t)\,dt}{x-x_0}\right).$$

Now $p(t)$ is nondecreasing, which gives

$$r(x) = \frac{\int_{x_0}^{x} p(t)\,dt}{x - x_0} \text{ is nondecreasing also.}$$

Thus, for $x_0 \le x \le c$, we have

$$\int_{y_1}^{B} \frac{z dz}{g(z)(B-z)^{\delta_2}} \ge \left(\int_{y_0}^{A} h(w)(A-w)^{\delta_1}\,dw\right) \cdot \left(\frac{\int_{x_0}^{c} p(t)\,dt}{c - x_0}\right)$$

$$\ge \left(\int_{y_0}^{A} h(w)(A-w)^{\delta_1}\,dw\right) \cdot p(x_0),$$

a contradiction to our assumption. This proves (a).

To prove (b), suppose that $\lim_{x \to c^-} y(x) \neq A.$ Then

$$\int_{x_0}^{x} \frac{y'(t)\,y''(t)}{g(y'(t))(B - y'(t))^{\delta_2}}\,dt = \int_{x_0}^{x} p(t)\,h(y(t))(A - y(t))^{\delta_1}\,y'(t)\,dt$$

which gives

$$\int_{y_1}^{y(x)} \frac{z dz}{g(z)(B-z)^{\delta_2}} \le p(x)\int_{y_0}^{y(x)} h(w)(A-w)^{\delta_2}\,dw$$

$$\le p(c_u)\int_{y_0}^{A} h(w)(A-w)^{\delta_2}\,dw.$$

Let $x \to c^-$. Then

$$\int_{y_0}^{A} \frac{z dz}{g(z)(B-z)^{\delta_2}} \le p(c_u)\int_{y_0}^{A} h(w)(A-w)^{\delta_2}\,dw,$$

a contradiction. This proves (b). □

Theorem 14. *Suppose* $\delta_1 \le -1$ *instead in* IVP (3.20). *Then blowup occurs* $\lim_{x \to c^-} y'(x) = B.$

Proof. First, assume $\delta_1 < -1$ and proceed as in the proof of Theorem 13.

Suppose $\lim_{x\to c^-} y'(x) \neq B$. Since both y and y' are nondecreasing, the Gruss inequality gives, for $x_0 \leq x \leq c$:

$$\int_{x_0}^{x} \frac{y'(t)\,y''(t)}{g(y'(t))\cdot(B - y'(t))^{\delta_2}}\,dt = \int_{x_0}^{x} p(t)\,h(y(t))\,(A - y(t))^{\delta_1}\,y'(t)\,dt.$$

So

$$\int_{y_1}^{y'(x)} \frac{zdz}{g(z)(B - z)^{\delta_2}}\,dt \geq \left(\int_{y_0}^{y(x)} h(w)(A - w)^{\delta_1}\,dt\right)\cdot\left(\frac{\int_{x_0}^{x} p(t)\,dt}{x - x_0}\right).$$

Since $B \geq y'(x)$ and the integrand is positive, we get

$$\int_{y_1}^{B} \frac{zdz}{g(z)(B - z)^{\delta_2}} \geq \left(\int_{y_0}^{y(x)} h(w)(A - w)^{\delta_1}\,dt\right)\cdot\left(\frac{\int_{x_0}^{x} p(t)\,dt}{x - x_0}\right).$$

Since $\lim_{x\to c^-} y(x) = A$, we obtain, upon letting $x \to c^-$:

$$\int_{y_1}^{B} \frac{zdz}{g(z)(B - z)^{\delta_2}} \geq \left(\int_{y_0}^{A} h(w)\cdot(A - w)^{\delta_1}\,dw\right)\cdot\frac{\int_{x_0}^{c} p(t)\,dt}{c - x_0}.$$

The integral on the left exists since $\delta_2 < 0$. However, the first integral on the right does not exist, because $\delta_1 < -1$, a contradiction to our assumption. The case $\delta_1 = -1$ is similar, since the integral again will not exist. This completes the proof. □

We now present a few numerical examples to illustrate the theorems of this section.

Example 2. Consider the IVP

$$y'' = (1 + x^2)\,y(1 - y)^{-1/2}(5 - y')^{-2},$$

$$y(0) = \frac{1}{10}, \quad y'(0) = 2.$$

Let

$$P_1 = \int_{y_0}^{A} h(w)(A - w)^{\delta_1} dw,$$

$$P_2 = \int_{y_1}^{B} \frac{zdz}{g(z)(B - z)^{\delta_2}}.$$

The Runge-Kutta fourth order method found

$$c \approx 0.4462, \quad y(c) = 1.0, \quad y'(c) = 2.0843.$$

So blowup occurs because $\lim_{x \to c^-} y(x) = A = 1$. We also have $P_1 = 1.3282$, $P_2 = 24.75$, $C^{**} = C_u = 7.274$ from (3.22). Then (3.21) of Lemma 12 fails to hold so part (b) of Theorem 13 is not applicable here. (C^* is much worse than C^{**}.) However, in the next two examples we shall see that Theorems 13 and 14 will predict a priori the nature of the blowup.

Example 3. Consider the IVP

$$y'' = (1 + x^2) y(5 - y)^{-1/2}(1 - y')^{-2},$$

$$y(0) = \frac{1}{10}, \quad y'(0) = \frac{1}{10}.$$

Then Runge-Kutta fourth order method finds

$$c \approx 1.3808, \quad y(c) = 0.4081, \quad y'(c) = 0.99993,$$

so blowup occurs because $\lim_{x \to c^-} y'(x) = B = 1$. This is verified by Theorem 13, part (a) since $P_1 = 14.9049$, $P_2 = 0.07898$ and $P(x_0)P_1 > P_2$ holds since $P(x_0) = P(0) = 1$ and $P_1 > P_2$.

Example 4. Consider the IVP

$$y'' = (1 + x^2) y(5 - y)^{-3/2}(1 - y')^{-2},$$

$$y(0) = \frac{1}{10}, \quad y'(0) = \frac{1}{10}.$$

Then Runge-Kutta finds $c \approx 2.4303$, $y(c) = 0.5738$, and $y'(c) = 0.99991$. Thus blowup occurs because $\lim_{x\to c} y'(x) = B = 1$. Thus, Theorem 14 correctly predicts the blowup nature, since $\delta_1 = -3/2 \le -1$ holds.

If the integrals defining P_1 and P_2 are not easily found, then the following result, which does not require integration, may be useful.

Theorem 15. *Suppose* p, h *and* g *in* IVP (3.20) *are nondecreasing functions of* x, y *and* y', *respectively. Suppose* $-1 < \delta_1 < 0$ *and* $\delta_2 < 0$. *Suppose there exists a real number* θ *with* $\theta \ge 0$ *and* $\theta + \delta_1 \le 0$, $y_1 > \frac{B}{2}$ *and*

$$0 = (-\theta)(B - y_1)(A - y_0)^{\delta_1} y_1$$
$$+ (A - y_0)^{\theta+\delta_1} p(x_0) h(y_0) g(y_1) (B - y_1)^{\delta_2} \ge 0. \tag{3.26}$$

Then blowup occurs because $\lim_{x\to c^-} y'(x) = B$.

Proof. Let $w = w(x) = \frac{(A - y)^\theta}{B - y'}$, $x_0 \le x < c$. Then

$$w' = w'(x) = (R_1(x) + R_2(x)) \cdot R_3(x),$$

where

$$R_1(x) = (B - y')\theta(A - y)^{\theta-1}(-y'),$$

$$R_2(x) = phg(A - y)^{\theta+\delta_1}(B - y')^{\delta_2}$$

and

$$R_3(x) = (B - y')^{-2}.$$

Since $0 \le \theta \le -\delta_1$ and $-1 < \delta_1 < 0$, we have $\theta - 1 \le 0$. $y_1 > \frac{B}{2}$, $R_1(x)$ is nondecreasing in x. Clearly, $R_2(x)$ and $R_3(x)$ are also nondecreasing in x. By (3.21), $w'(x_0) \ge 0$. Since $w'(x) \ge 0$, we have $w(x)$ is nondecreasing in x. Since $w(x_0) \ge 0$, $w(x)$ never approaches zero as $x \to c^-$. Thus, blowup occurs in (3.20) because $\lim_{x\to c^-} y'(x) = B$, as claimed. □

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