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Non-chaotic behaviour in three-dimensional quadratic systems

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Abstract. It is shown that three-dimensional dissipative quadratic systems of ordinary differential equations with a total of four terms on the right-hand side of the equations do not exhibit chaos. This complements recent work of Sprott who has given many examples of chaotic quadratic systems with as few as five terms on the right-hand side of the equations.

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1. Introduction

How complicated must an ordinary differential equation be in order to exhibit chaotic behaviour? The Poincaré–Bendixson theorem shows that chaos does not exist in a two-dimensional autonomous system (or second-order equation) [9]. The three-dimensional Lorenz equations [4],

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y\\ \dot{y} &= rx - xz - y\\ \dot{z} &= xy - bz\end{aligned}$$

do exhibit chaos for certain values of the parameters σ, b, and r. So does the Rossler system [6],

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + xz - cz\end{aligned}$$

again for certain parameter values. Likewise for a number of other three-dimensional systems [2, 3].

Very interesting investigations have recently been carried out by Sprott [7, 8] raising the question as to whether a total of seven terms on the right-hand side of a three-dimensional system is really necessary. Sprott first performed a computer search on the entire class of three-dimensional quadratic systems and found numerous cases of chaos in systems with

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six terms on the right-hand side with only one nonlinear (quadratic) term. He also found numerous examples of chaotic five-term systems with two nonlinear terms. In a follow-up study Sprott examined five-term systems with only one nonlinear term and found two examples of chaotic systems. No chaotic systems were found with either just three or four terms with no limit on the number of (quadratic) nonlinearities.

Sprott's work raises an obvious question: What is the behaviour of solutions of three-dimensional systems when there are less than five terms? If such systems cannot exhibit chaos, why is this so? The purpose of this paper is to examine these questions. We are able to resolve the issue for all dissipative four-term equations (three-term equations are trivial in this respect). Our methods may extend to include all four-term equations which are not necessarily dissipative. The increase in complexity is thereby non-trivial and is discussed in section 7. Thus, we restrict our attention to dissipative systems partly for convenience but also because it is dissipative systems which are most likely to arise in applications. Also it is for dissipative systems where chaos, when it exists, has one of its most graphic manifestations, the strange attractor.

It turns out that the most complicated four-term three-dimensional dissipative systems (which are neither integrable nor reducible to two-dimensional systems) exhibit only two different types of behaviour. Solutions are either asymptotic to a two-dimensional surface or they have a limit (that is, converge to an equilibrium point) which may be infinite. Neither type of behaviour is chaotic. This paper is largely based on the thesis of Zhang [10] which also contains other approaches which will be developed in future work.

The plan of the paper is as follows. After quickly disposing of three-term systems, we take up in turn, four-term equations with either one, two, or three nonlinear terms, all without constant terms. Systems with four nonlinear terms cannot be dissipative. Then equations with constant terms are considered in a separate section.

In each section we begin by listing all possible equations in the appropriate category which are not permutationally equivalent to each other, nor which reduce to two-dimensional systems or linear systems. These lists of equations were obtained by the 'brute force' method of listing all possible combinations of variables in a systematic way and then simply pulling out all of the non-trivial dissipative systems. The reader can easily reconstruct these exhaustive lists of equations although the process is tedious.

Our analysis of establishing the two basic types of non-chaotic behaviour shows in every case that a particular equation has only these two types of behaviour. However, some equations may have both types of behaviour simultaneously and thus further analysis, if even possible, is required to determine when this dichotomy occurs. It will depend, of course, on dividing the three-dimensional (x, y, z) space into different subregions of initial conditions for each of the two different types of behaviour.

2. Three-term dissipative systems

We mention in passing that three-term three-dimensional dissipative systems are trivially non-chaotic. Typical examples are

$$\begin{cases} \dot{x} = -x \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases}$$

which is solvable [5] and both

$$\begin{cases} \dot{x} = -x \\ \dot{y} = xz \\ \dot{z} = xy \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{x} = -x \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases}$$

which reduce to two-dimensional autonomous systems.

3. Four-term dissipative systems with one nonlinearity

Consider the $4-1$ case for which a typical example is

$$\begin{cases} \dot{x} = ayz - bx \\ \dot{y} = cx \\ \dot{z} = dy. \end{cases} \qquad b > 0 \tag{3.1}$$

We now show that all four parameters can be eliminated by making the transformation $x' = \alpha x$, $y' = \beta y$, $z' = \gamma z$, $t' = \delta t$. Thus

$$\dot{x} = \frac{\delta}{\alpha}\frac{\mathrm{d}x'}{\mathrm{d}t'} \qquad \dot{y} = \frac{\delta}{\beta}\frac{\mathrm{d}y'}{\mathrm{d}t'} \qquad \dot{z} = \frac{\delta}{\gamma}\frac{\mathrm{d}z'}{\mathrm{d}t'}$$

which gives the new system

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t'} = \dfrac{a\alpha}{\delta\beta\gamma}y'z' - \dfrac{b}{\delta}x' \\ \dfrac{\mathrm{d}y'}{\mathrm{d}t'} = \dfrac{c\beta}{\delta\alpha}x' \\ \dfrac{\mathrm{d}z'}{\mathrm{d}t'} = \dfrac{\mathrm{d}\gamma}{\delta\beta}y'. \end{cases}$$

We now set $\frac{a\alpha}{\delta\beta\gamma} = \frac{b}{\delta} = \frac{c\beta}{\delta\alpha} = \frac{d\gamma}{\delta\beta} = 1$, thus $\delta = b > 0$ (so that time is not reversed) and we can solve to find

$$\beta = \frac{ac}{b^2} \qquad \alpha = \frac{ac^2}{db^2} \qquad \gamma = \frac{ac}{bd}.$$

Thus (3.1) is transformed into

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t'} = y'z' - x' \\ \dfrac{\mathrm{d}y'}{\mathrm{d}t'} = x' \\ \dfrac{\mathrm{d}z'}{\mathrm{d}t'} = y' \end{cases}$$

and all four arbitrary parameters a, b, c, d are removed by rescaling. All $4-1$ equations can be rescaled in the same way. A complete list of the dissipative equations in the $4-1$ case (eliminating equivalent, two-dimensional, and linearly reducible systems) is:

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = z \\ \dot{z} = x \end{cases} \tag{3.2}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x \\ \dot{z} = -z \end{cases} \tag{3.3}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x \\ \dot{z} = y \end{cases} \tag{3.4}$$

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = x + z \\ \dot{z} = -z \end{cases} \tag{3.5}$$

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = z - y \\ \dot{z} = x \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = z \\ \dot{z} = x - z \end{cases} \tag{3.7}$$

$$\begin{cases} \dot{x} = yz \\ \dot{y} = x \\ \dot{z} = x - z \end{cases} \tag{3.8}$$

$$\begin{cases} \dot{x} = yz \\ \dot{y} = x \\ \dot{z} = y - z. \end{cases} \tag{3.9}$$

Theorem. *None of the systems (3.2)–(3.9) are chaotic.*

Proof. For (3.2) $\dot{y} = z$, $\ddot{y} = \dot{z} = x$, and $\dddot{y} = \dot{x} = y^2 - x = y^2 - \ddot{y}$. This third-order scalar equation integrates to

$$\ddot{y} + \dot{y} = c + \int_0^t (y(s))^2 \, \mathrm{d}s.$$

Thus $\ddot{y}(t) + \dot{y}(t)$ is monotone increasing and has a limit $L \leqslant \infty$. If $L < \infty$, then $\ddot{y}(t) + \dot{y}(t) \to S$ where S is the two-dimensional surface (in $\{(x^1, x^2, x^3)\} = R^3$ phase space) $x^3 + x^2 - L = 0$. Thus, any attractor for $y(t)$ is two dimensional and therefore not chaotic. If $L = \infty$, then $\dot{y}(t) \to \infty$ as $t \to \infty$ and hence $y(t)$, $z(t)$, $x(t)$ all $\to \infty$ as $t \to \infty$. Thus all three components of (3.2) have a limit as $t \to \infty$ and so the solution is not chaotic.

For (3.3) a similar process yields

$$\ddot{y}(t) = y^2(t) + c\mathrm{e}^{-t}.$$

Since $\ddot{y}(t) - y^2(t) \to 0$ as $t \to \infty$, any attractor for $y(t)$ is a solution of $\ddot{u} = u^2$ and hence two-dimensional.

For (3.4) the scalar equation is $\dddot{z} + \ddot{z} - \dot{z}z = 0$ which can be integrated to a second-order equation and thus is non-chaotic. For (3.5) the scalar equation $\ddot{y}(t) = y^2(t) + c\mathrm{e}^{-t}$ is obtained, exactly the same as (3.3). For (3.6) the scalar equation is $\dddot{y} + \ddot{y} = y^2$, the same as for (3.2). For (3.7), the scalar equation is again $\dddot{y} + \ddot{y} = y^2$. For (3.8), the scalar equation is $\dddot{y}y + \ddot{y}y = \dot{y}\ddot{y} + y^2\dot{y}$ which integrates to

$$y\ddot{y} + y\dot{y} - \frac{\dot{y}^2}{2} - \frac{y^3}{3} = c + \int_0^t (\dot{y}(s))^2\,\mathrm{d}s.$$

Thus $y\ddot{y}+y\dot{y}-\frac{\dot{y}^2}{2}-\frac{y^3}{3} \to L \leqslant \infty$ as $t \to \infty$. If $L < \infty$, the attractor is a two-dimensional surface. If $L = \infty$ then $y(t) \to -\infty$ as $t \to \infty$. $\dot{z} = x - z$ leads to

$$\ddot{z} + \dot{z} + (-y)z = 0$$

which (since $-y(t) \to \infty$) is a super critical damped second-order linear equation and so $z(t)$, $\dot{z}(t) \to 0$ as $t \to \infty$. Thus $x(t) \to 0$ as $t \to \infty$ and (x, y, z) has a limit as $t \to \infty$.

For (3.9) the scalar equation is $\dddot{z} + \ddot{z} = z\dot{z} + z^2$ which integrates to

$$\ddot{z} + \dot{z} - \frac{z^2}{2} = c + \int_0^t (z(s))^2\,\mathrm{d}s$$

and $\ddot{z}(t) + \dot{z}(t) - \frac{(z(t))^2}{2} \to L \leqslant \infty$ as $t \to \infty$. If $L < \infty$, a two-dimensional surface is obtained as usual. If $L = \infty$ then $z(t) \to \pm\infty$ as $t \to \infty$. If $z(t) \to -\infty$, $\dot{z}(t) \to \infty$, a contradiction. If $z(t) \to \infty$, then $\dot{z}(t) \to \infty$ and $y(t) \to \infty$. Thus $\dot{x}(t) \to \infty$ and so $x(t) \to \infty$, (x, y, z) has a limit and there is no chaos. □

4. Four-term dissipative systems with two nonlinearities

Now turning to the case of four-term dissipative systems with two nonlinearities, we again begin by discussing what happens when the equations are rescaled to eliminate the arbitrary parameters. This time consider the typical example

$$\begin{cases} \dot{x} = ayz - bx \\ \dot{y} = cxz & \qquad b > 0 \\ \dot{z} = dy. \end{cases}$$

The same substitution $x' = \alpha x,\ y' = \beta y,\ z' = \gamma z,\ t' = \delta t$ leads to

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t'} = \dfrac{\alpha a}{\delta\beta\gamma} y'z' - \dfrac{b}{\delta} x' \\ \dfrac{\mathrm{d}y'}{\mathrm{d}t'} = \dfrac{c\beta}{\delta\alpha\gamma} x'z' \\ \dfrac{\mathrm{d}z'}{\mathrm{d}t'} = \dfrac{d\gamma}{\delta\beta} y'. \end{cases}$$

We again take $\delta = b > 0$ and time is not reversed. Setting the other coefficients equal to 1 and solving, we obtain

$$\alpha = \frac{dc}{b^2} \qquad \beta^2 = \frac{acd^2}{b^4} \qquad \gamma^2 = \frac{ac}{b^2}.$$

This requires that $ac > 0$, or conversely we can only rescale to 1 within a $\pm$ sign for one of the nonlinear terms. The above system is thus rescaled to

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t'} = y'z' - x' \\ \dfrac{\mathrm{d}y'}{\mathrm{d}t'} = \pm x'z' \\ \dfrac{\mathrm{d}z'}{\mathrm{d}t'} = y' \end{cases}$$

placing the $\pm$ sign on the equation with only one term. The analysis of the different $4-2$ cases may be affected by a $\pm$ sign on one of the terms. Thus, both signs are indicated below when the minus sign cannot be transformed away. The $4-2$ cases can now be listed (eliminating equivalent, linear and two-dimensional systems):

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x \\ \dot{z} = -z \end{cases} \tag{4.1}$$

$$\begin{cases} \dot{x} = y^2 + z^2 \\ \dot{y} = x \\ \dot{z} = -z \end{cases} \tag{4.2}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = xz \\ \dot{z} = \pm y \end{cases} \tag{4.3}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = xz \\ \dot{z} = kz \end{cases} \qquad k < 1 \tag{4.4}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = z^2 \\ \dot{z} = x \end{cases} \tag{4.5}$$

$$\begin{cases} \dot{x} = y^2 + y \\ \dot{y} = xz \\ \dot{z} = -z \end{cases} \tag{4.6}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = x^2 \\ \dot{z} = -z \end{cases} \tag{4.7}$$

$$\begin{cases} \dot{x} = y^2 + z \\ \dot{y} = xz \\ \dot{z} = -z \end{cases} \tag{4.8}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x^2 \\ \dot{z} = \pm x \end{cases} \tag{4.9}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x^2 \\ \dot{z} = y \end{cases} \tag{4.10}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x^2 \\ \dot{z} = kz \end{cases} \qquad k < 1 \tag{4.11}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = \pm xz \\ \dot{z} = y \end{cases} \tag{4.12}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = z^2 \\ \dot{z} = \pm x \end{cases} \tag{4.13}$$

$$\begin{cases} \dot{x} = \pm y - x \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases} \tag{4.14}$$

$$\begin{cases} \dot{x} = \pm y - x \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{4.15}$$

$$\begin{cases} \dot{x} = y - x \\ \dot{y} = z^2 \\ \dot{z} = x^2 \end{cases} \tag{4.16}$$

$$\begin{cases} \dot{x} = y - x \\ \dot{y} = z^2 \\ \dot{z} = xy. \end{cases} \tag{4.17}$$

Theorem. *Systems (4.1)–(4.17) are not chaotic.*

Proof. System (4.1) reduces to the scalar equation $\ddot{y}(t) = (y(t))^2 + c\mathrm{e}^{-t}y(t)$. It is clear that if $y(t)$ is unbounded then $\lim_{t\to\infty} y(t) = \pm\infty$, $\lim_{t\to\infty} \ddot{y}(t) = \infty$ and thus $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \dot{x}(t) = \infty$. Thus (x, y, z) has a limit as $t \to \infty$. On the other hand, if $y(t)$ is bounded then $\ddot{y}(t) - y^2(t) \to 0$ and thus the attractor for $y(t)$ is on a two dimensional surface. So there is no chaos.

System (4.2) reduces to $\ddot{y}(t) = y^2(t) + c\mathrm{e}^{-2t}$ and thus $\ddot{y}(t) - y^2(t) \to 0$ as $t \to \infty$ and any attractor for $y(t)$ is two dimensional.

For system (4.3) with a plus sign we obtain

$$z\ddot{z} - \dot{z}^2 + z\dot{z} = c + \int_0^t (\dot{z}(s))^2\,\mathrm{d}s + \int_0^t z^2(s)(\dot{z}(s))^2\,\mathrm{d}s$$

which has a monotone left-hand side. Now consider the minus sign case which is quite complicated. Note that either $x(t)$ is negative and increasing or x becomes and remains positive.

Case 1. $\boldsymbol{x(t) \uparrow L \leqslant 0}$ ($\boldsymbol{x(t)}$ is monotone increasing to $\boldsymbol{L}$). Here the argument is similar to previous cases. First suppose $z_0 > 0$. Thus $\dot{y}$ starts out negative. If $y_0 > 0$, then $\dot{z}$ starts out negative. If y becomes negative first, then z remains positive and hence y remains negative. Hence $y(t) \downarrow K$ and $z(t) \uparrow M$ as $t \to \infty$. If z becomes negative first then y remains positive, z remains negative, and $y(t) \uparrow K$, $z(t) \downarrow M$ as $t \to \infty$. If $y_0 < 0$ then $\dot{z}$ starts out positive, and $y(t) \downarrow K$, $z(t) \uparrow M$ as $t \to \infty$. Now suppose $z_0 < 0$ so that $\dot{y}$ starts out positive. If $y_0 > 0$ then $\dot{z}$ starts out negative; hence $y(t) \uparrow K$, $z(t) \downarrow M$ as $t \to \infty$. If $y_0 < 0$, then $\dot{z}$ starts out positive. If y becomes positive before z, then z remains negative, y remains positive and so $y(t) \uparrow K$, $z(t) \downarrow M$ as $t \to \infty$. If z becomes positive first, then y remains negative, z remains positive and so $y(t) \downarrow K$, $z(t) \uparrow M$ as $t \to \infty$.

Case 2. $x(t)$ is positive for $t > 0$. We first establish the following.

Lemma.

$$\frac{\dot{y}^2}{x^2} + \frac{y^2}{x^2} + \ln x = c - \int_0^t \left(\frac{\dot{x}(s)}{x(s)}\right)^2 \mathrm{d}s \downarrow L.$$

Proof. Since $\dot{z} + y = 0$ and $z = \frac{\dot{y}}{x}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{x}\right) + y = 0.$$

Thus we can write

$$\frac{\dot{y}}{x}\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\dot{y}}{x}) + \frac{y\dot{y}}{x} = 0$$

or

$$\frac{1}{2}\left(\frac{\dot{y}}{x}\right)^2 + \frac{y^2}{2x} + \int_0^t \frac{y^2(s)}{2}\frac{\dot{x}(s)}{x^2(s)}\,\mathrm{d}s = c$$

or

$$\frac{\dot{y}^2}{x^2} + \frac{y^2}{x} = c - \int_0^t \frac{\dot{x}(s)}{x^2(s)}(\dot{x}(s) + x(s))\,\mathrm{d}s$$

or

$$\frac{\dot{y}^2}{x^2} + \frac{y^2}{x} + \ln x = c - \int_0^t \left(\frac{\dot{x}(s)}{x(s)}\right)^2 \mathrm{d}s$$

which proves the lemma. □

If $L = -\infty$, then clearly $x(t) \to 0$ as $t \to \infty$. But $|\ln x|/x \to \infty$ as $x \downarrow 0$ and thus also $y(t) \to 0$ as $t \to \infty$. Since $\ddot{z} + xz = 0$, $z(t)$ is either monotone for large t or oscillatory, depending on how quickly $x(t) \to 0$ [1]. Even if it is oscillatory, $z(t)$ is asymptotic to the line $x = 0$, $y = 0$.

If $L > -\infty$, then writing $y^2 = \dot{x} + x$ and $\dot{y}^2 = (\ddot{x} + \dot{x})^2/4(\dot{x} + x)^2$, the above expression becomes

$$\frac{(\ddot{x} + \dot{x})^2}{4x^2(\dot{x} + x)^2} + \frac{\dot{x} + x}{x} + \ln x \downarrow L$$

and, as before, the solution $x(t)$ is asymptotic to a two-dimensional surface in phase space and therefore non-chaotic.

System (4.4) has the scalar form $\ddot{y}(t) + (1 - k)\dot{y}(t) = c\mathrm{e}^{kt}(y(t))^2$. Thus either $y(t)$ has a two-dimensional attractor or $y(t) \to \pm\infty$ as $t \to \infty$. Then $\lim_{t\to\infty} x(t) = \infty$ also so (x, y, z) has a limit as $t \to \infty$.

Because of the two squared terms, system (4.5) is easy to resolve. Since $y(t)$ is increasing we have $y(t) \to L \leqslant \infty$. If $L \leqslant 0$, $\dot{x}(t) + x(t) \to L^2$ and $x(t) \to L^2$. Thus $\dot{z} \geqslant 0$ and $z(t)$ has a limit as $t \to \infty$. Since all three components (x, y, z) have limits at ∞, there is no chaos. If $L > 0$, the argument is similar.

System (4.6) has the scalar form $\ddot{y}(t) + \dot{y}(t) = c\mathrm{e}^{-t}(y(t))^2 + c\mathrm{e}^{-t}y(t)$ or $\mathrm{e}^t\ddot{y}(t)\dot{y}(t) + \mathrm{e}^t(\dot{y}(t))^2 = c(y(t))^2\dot{y}(t) + cy(t)\dot{y}(t)$ which integrates to

$$c\frac{(y(t))^3}{3} + c\frac{(y(t))^2}{2} - \mathrm{e}^t\frac{(\dot{y}(t))^2}{2} = c_1 + \int_0^t \mathrm{e}^s\frac{(\dot{y}(s))^2}{2}\,\mathrm{d}s$$

which means that $\frac{c}{3}(y(t))^3 + \frac{c}{2}(y(t))^2 - \frac{\mathrm{e}^t}{2}(\dot{y}(t))^2 \uparrow L \leqslant \infty$ as $t \to \infty$. If $L = \infty$ then $y(t) \to \pm\infty$ as $t \to \infty$ depending upon whether c is positive or negative. Thus $x(t) \to \infty$

as $t \to \infty$ and so (x, y, z) has a limit at $t = \infty$. If $L < \infty$, then the attractor for y lies on the two-dimensional surface determined by

$$\frac{c}{3}(y(t))^3 + \frac{c}{2}(y(t))^2 - \frac{\mathrm{e}^{-t}}{2}(\dot{y}(t))^2 = L$$

(recall that a first-order non-autonomous equation is equivalent to a second-order autonomous equation).

For system (4.7) $z(t) = c\mathrm{e}^{-t} \to 0$ as $t \to \infty$ and $y(t) \to L \leqslant \infty$ as $t \to \infty$. If $L \neq 0$ then $x(t) \to \infty$ as $t \to \infty$. Suppose $L = 0$, then $\lim_{t\to\infty} \dot{x}(t) = 0$ and any attractor for the system lies on a surface $x(t) =$ constant.

System (4.8) can be rewritten as the scalar equation $\ddot{y}(t) + \dot{y}(t) = c\mathrm{e}^{-t}(y(t))^2 + c^2\mathrm{e}^{-2t}$ which integrates to

$$\dot{y}(t) + y(t) + \frac{c^2}{2}\mathrm{e}^{-2t} = c_1 + c\int_0^t \mathrm{e}^{-s}(y(s))^2\,\mathrm{d}s.$$

Thus $\dot{y}(t)+y(t) \to L$ as $t \to \infty$. If $L = \pm\infty$, then $y(t) \to \pm\infty$ as $t \to \infty$ and $x(t) \to \infty$ as $t \to \infty$. If L is finite the attractor for $y(t)$ lies on a one-dimensional surface.

System (4.9) with the plus sign leads to

$$z\ddot{z} - z^2 + z\dot{z} = c + 2\int^t \dot{z}^2(s)\,\mathrm{d}s + \int^t z^2(s)\dot{z}^2(s)\,\mathrm{d}s \uparrow L$$

which is treated in the now familiar way. With the minus sign, the above argument breaks down and a different approach is needed. Observe first that y is increasing.

Case 1. $y \uparrow L \leqslant 0$. First suppose $x_0 = x(0) > 0$. Then $\dot{z}$ starts out negative. If $z_0 > 0$ then $\dot{x}$ also starts out negative. If x becomes negative before z then $\dot{z}$ becomes positive and z remains positive. Since x, once negative, remains negative, then $\dot{x}$ remains negative, $\dot{z}$ remains positive and hence (x, y, z) all have limits as $t \to \infty$. Now suppose that z becomes negative before x, then x and $\dot{x}$ remain positive, hence $\dot{z}$ remains negative and again (x, y, z) all have limits as $t \to \infty$. If $z_0 < 0$ then $\dot{z} < 0$ and $z < 0$ as long as $x > 0$. But x cannot become negative. Hence $\dot{z}$ remains negative, $z(t) \downarrow K$ and $x(t) \to LK$ or $t \to \infty$. If $x_0 < 0$, similar arguments show that (x, y, z) all have limits as $t \to \infty$.

Case 2. $y(t) \uparrow L > 0$. Then $\ddot{z}+\dot{z}+yz = 0$ which is essentially a positively damped linear harmonic oscillator. Thus $z(t)$, $\dot{z}(t) \to 0$ as $t \to \infty$, and hence also $x(t) \to 0$ as $t \to \infty$.

For system (4.10) there is only one sign to consider and the discussion is brief. $y(t) \uparrow L$ and $t \to \infty$. If $L \leqslant 0$ then $z(t) \downarrow K$ and hence $x(t) \to LK$ as $t \to \infty$. If $L > 0$, then $z(t) \uparrow K$ and $x(t) \to LK$ as $t \to \infty$.

System (4.11) has the scalar form $\ddot{x}(t) + (2-k)\dot{x}(t) - kx(t) = c\mathrm{e}^{kt}(x(t))^2$. Since the linear homogeneous part of this equation has a characteristic equation with real roots, then use of the variation of parameters formula shows that all solutions $x(t)$ of the nonlinear equation have a limit at ∞. Since $y(t)$ is non-decreasing for all t, then (x, y, z) has a limit as $t \to \infty$.

System (4.12) can be rewritten in scalar form as $z\dddot{z}+z\ddot{z}-\dot{z}\ddot{z}\mp z^3\dot{z} = 0$ which integrates to

$$z\ddot{z} - \dot{z}^2 + z\dot{z} \mp \frac{z^4}{4} = c \pm \int_0^t z^2(s)\,\mathrm{d}s \uparrow L \leqslant \infty.$$

If $L < \infty$, the solution is asymptotic to a two-dimensional surface and there is no chaos. Clearly $L = \infty$ is impossible.

Consider system (4.13) with the plus sign. We obtain

$$z\ddot{z} - \dot{z}^2 + z\dot{z} = c + 2\int_0^t \dot{z}^2(s)\,\mathrm{d}s + \int_0^t z^4(s)\,\mathrm{d}s$$

and the usual monotonicity argument applies. For the minus sign, the equation is almost identical to (4.2) and the same argument applies to show that (x, y, z) all have limits.

For system (4.14) with a plus sign we obtain

$$x\dot{x} - \dot{x}^2 + x\dot{x} = c + 2\int_0^t (\dot{x}(s))^2\,\mathrm{d}s + \int_0^t x^4(s)\,\mathrm{d}s$$

for which the left-hand side is monotone increasing. With a minus sign in (4.14) a similar argument will apply.

For system (4.15) with the plus sign we obtain

$$x\ddot{x} - \dot{x}^2 + x\dot{x} = c + \int_0^t (\dot{x}(s))^2\,\mathrm{d}s + \int_0^t x^2(s)(\dot{x}(s) + x(s))^2\,\mathrm{d}s$$

and so again the monotonicity of the left-hand side eliminates the possibility of chaos. With a minus sign the above argument breaks down and so we proceed as follows. z is monotone increasing to L.

Case 1. $L \leqslant 0$. Suppose first that $x_0 > 0$. Then $\dot{y}$ starts out negative. If $y_0 > 0$ then $\dot{x}$ also starts out negative. If x becomes negative before y then x remains negative and y remains positive. Thus $y(t)$ increases to K, $x(t) \to -K$ and so (x, y, z) all have limits as $t \to \infty$. If y becomes negative before x, then x must remain positive. Hence $y(t)$ again decreases to K, $x(t) \to -K$ and (x, y, z) all have limits as $t \to \infty$. If $y_0 < 0$, then both y and $\dot{y}$ remain negative since x must remain positive. Again $y(t) \downarrow K$, and $x(t) \to -K$ as $t \to \infty$. Now suppose that $x_0 < 0$. Then $\dot{y}$ starts out positive. If $y_0 > 0$, then both x and y remain positive, $y(t) \uparrow K$, $x(t) \to -K$ as $t \to \infty$. If $y_0 < 0$ then $\dot{x}(t)$ starts out positive. If x becomes positive before y then y remains negative and x remains positive. Thus $y(t) \uparrow K$, $x(t) \to -K$ as $t \to \infty$. If y becomes positive before x, then x remains negative, hence y remains positive, $y(t) \uparrow K$, $x(t) \to -K$ as $t \to \infty$.

Case 2. $L > 0$. Then we obtain $\ddot{x} + \dot{x} + zx = 0$ with $z(t) \uparrow L > 0$. Again we have a positively damped harmonic oscillator and so $x(t)$, $\dot{x}(t) \to 0$ as $t \to \infty$. Thus also $y(t) \to 0$.

For system (4.16) there is only one sign and two squared terms. Hence $z(t) \uparrow L$, $y(t) \uparrow K$ and $x(t) \to K$ and so (x, y, z) has a limit as $t \to \infty$.

Finally, for system (4.17) $\dot{y} \geqslant 0$ and so $y(t) \to L$ as $t \to \infty$. Thus also $x(t) \to L$ as $t \to \infty$. If $L \neq 0$ then $z(t) \to \infty$ as $t \to \infty$ and so $L = \infty$. Now suppose that $L = 0$. Then $y(t) < 0$ for all t. $x_0 = x(0) > 0$. If $x(t) \geqslant 0$ for all $t \geqslant 0$ then $\dot{z}(t) \leqslant 0$ for all $t \geqslant 0$ and $z(t)$ has a limit as $t \to \infty$. Suppose $x(t_0) = 0$ for some t_0. Then $\dot{x}(t_0) < 0$ and $x(t)$ becomes negative. Since $\dot{x}(t) < 0$ for $x(t) > y(t)$ then $x(t_1) = y(t_1)$ for some $t_1 > t_0$. Then $x(t) \leqslant y(t) < 0$ for $t \geqslant t_1$. Thus $\dot{z}(t) > 0$ for $t > t_0$ and $z(t)$ has a limit at ∞. If $x_0 = x(0) \leqslant 0$ a similar argument applies. Thus (x, y, z) all have the limit 0 or ∞ as $t \to \infty$.

This disposes of all $4 - 2$ dissipative cases and proves the theorem for this section.

5. Four-term dissipative systems with three nonlinear terms

The next case is the four-term dissipative systems with three nonlinear terms. The four arbitrary parameters are once again removed by the scaling transformation $x' = \alpha x$, $y' = \beta y$,

$z' = \gamma z$, $t' = \delta t$. Again in certain cases we can only rescale to 1 within a $\pm$ sign. The $4-3$ cases are (eliminating equivalent, two-dimensional, and essentially linear systems):

$$\begin{cases} \dot{x} = x^2 + yz \\ \dot{y} = -2xy \\ \dot{z} = -z \end{cases} \tag{5.1}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = x^2 \\ \dot{z} = -z \end{cases} \tag{5.2}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = \pm xz \\ \dot{z} = -z \end{cases} \tag{5.3}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = x^2 \\ \dot{z} = -z \end{cases} \tag{5.4}$$

$$\begin{cases} \dot{x} = y^2 \pm z^2 \\ \dot{y} = xz \\ \dot{z} = -z \end{cases} \tag{5.5}$$

$$\begin{cases} \dot{x} = xy - x \\ \dot{y} = xz \\ \dot{z} = -yz \end{cases} \tag{5.6}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = xz \\ \dot{z} = x^2 \end{cases} \tag{5.7}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{5.8}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = z^2 \\ \dot{z} = x^2 \end{cases} \tag{5.9}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = z^2 \\ \dot{z} = xy \end{cases} \tag{5.10}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x^2 \\ \dot{z} = \pm xy \end{cases} \tag{5.11}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x^2 \\ \dot{z} = y^2 \end{cases} \tag{5.12}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = \pm xz \\ \dot{z} = y^2 \end{cases} \tag{5.13}$$

$$\begin{cases} \dot{x} = y^2 + yz \\ \dot{y} = -y \\ \dot{z} = x^2. \end{cases} \tag{5.14}$$

Theorem. *Systems (5.1)–(5.14) are not chaotic.*

Proof. System (5.1) can be rewritten as the scalar equation $\ddot{x}+\dot{x} = 2x^3+x^2$ which, when multiplied by $\dot{x}$, integrates to

$$-\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 = c + \int_0^t (\dot{x}(s))^2\,\mathrm{d}s \uparrow L.$$

If $L = \infty$ then $\lim_{t\to\infty} x(t) = \pm\infty$. Thus $\lim_{t\to\infty} y(t) = \mp\infty$ and since $\lim_{t\to\infty} z(t) = 0$, (x, y, z) has a limit at $t = \infty$. If $L < 0$, then any attractor for $x(t)$ lies on a two-dimensional surface.

System (5.2) can be rewritten as the two-dimensional (non-autonomous) system, $\dot{x} = y^2 + c\mathrm{e}^{-t}y$, $\dot{y} = x^2$. Thus $\lim_{t\to\infty} y(t) = L \leqslant \infty$. If $L = \infty$, then also $\lim_{t\to\infty} x(t) = \infty$. If $L < \infty$ then $\lim_{t\to\infty} \dot{x}(t) = L^2$. If $L \neq 0$, then $\lim_{t\to\infty} x(t) = \infty$. If $L = 0$, then $\lim_{t\to\infty} \dot{x}(t) = 0$ and any attractor for $(x(t), y(t), z(t))$ lies on a two-dimensional surface $x =$ constant.

Consider system (5.3) with a plus sign and take $z(t) = c_1\mathrm{e}^{-t}$ where $c_1 > 0$. The equivalent scalar equation is $\ddot{y}(t) + \dot{y}(z) = c\mathrm{e}^{-t}(y(t))^2 + c^2\mathrm{e}^{-2t}(y(t))^2$. Thus if $y(t)$ is bounded, then $\lim_{t\to\infty}(\ddot{y}(t) + \dot{y}(t)) = 0$ and any attractor for $y(t)$ is two dimensional. Suppose $y(t)$ is unbounded and not eventually monotone. Rewrite (5.3) in the form

$$\begin{cases} \dot{x}(t) = (y(t))^2 + c\mathrm{e}^{-t}y(t) \\ \dot{y}(t) = cx(t)\mathrm{e}^{-t}. \end{cases}$$

Clearly $y(t)$ in this case can never become positive. Also $\dot{x}(t) \geqslant 0$ except for $0 > y(t) < -c\mathrm{e}^{-t}$. Thus $\limsup_{t\to\infty} y(t) = 0$ and $\liminf_{t\to\infty} y(t) = -\infty$. To have both $x(t) \leqslant 0$ and $\dot{x}(t) \leqslant 0$ requires $-c\mathrm{e}^{-t} \leqslant y(t) \leqslant 0$ and $\dot{y}(t) \leqslant 0$. If $y(t) \leqslant -c\mathrm{e}^{-t} < 0$, then $x(t) < 0$, $\dot{x}(t) > 0$, and $\ddot{y}(t) = c\mathrm{e}^{-t}\dot{x}(t) - c\mathrm{e}^{-t}x(t) > 0$. Thus $\liminf_{t\to\infty} y(t) = -\infty$ leads to a contradiction. If $z(t)$ is negative or (5.3) has a minus sign in the second equation, similar arguments can be made. Thus (5.3) has no chaotic behaviour.

If system (5.4) has a plus sign, then all three variables $x(t)$, $y(t)$, $z(t)$ are monotone and have a limit. Consider (5.4) with the minus sign which can be rewritten as

$$\begin{cases} \dot{x}(t) = (y(t))^2 - c^2\mathrm{e}^{-2t} \\ \dot{y}(t) = (x(t))^2. \end{cases}$$

Suppose $\lim_{t\to\infty} y(t) = L$. If $L \neq 0$, then $\lim_{t\to\infty} x(t) = \infty$. If $L = 0$, then $\lim_{t\to\infty} \dot{x}(t) = 0$ and so any attractor for (5.4) lies on a surface $x(t) =$ constant.

System (5.5) can be rewritten as the scalar equation $\ddot{y}(t) + \dot{y}(t) = c\mathrm{e}^{-t}(y(t))^2 \pm c^3\mathrm{e}^{-3t}$ which integrates to

$$\dot{y}(t) + y(t) = c_1 + \int_0^t c\mathrm{e}^{-s}(y(s))^2\,\mathrm{d}s \mp \frac{c^3}{3}\mathrm{e}^{-3t}.$$

Thus $\dot{y}(t) + y(t)$ has a limit L as $t \to \infty$. If $L = \pm\infty$, then $\lim_{t\to\infty} y(t) = \pm\infty$ and $\lim_{t\to\infty} x(t) = \infty$. If L is finite, then $\dot{y}(t) + y(t)$ converges to L as $t \to \infty$ and any attractor for $y(t)$ is two dimensional.

For system (5.6) the equivalent scalar equation is $\ddot{y}(t) + \dot{y}(t) = 0$ and thus (5.6) reduces to a linear system which cannot be chaotic.

For system (5.7) $\lim_{t\to\infty} z(t) = L \leqslant \infty$ and $x(t)$ is increasing when $x(t) < 0$. Thus either $\lim_{t\to\infty} x(t) = K$ exists or $x(t)$ is eventually positive. $y(t)$ is monotone in either case and hence has a limit as $t \to \infty$. Thus (x, y, z) all have limits at $t = \infty$.

Systems (5.8)–(5.10) and (5.12) are similar to system (5.7) and hence non-chaotic. System (5.11) leads to $x\dot{x} \mp z\dot{z} + \dot{y} = 0$ which is integrable. System (5.13) leads to $x\dot{x} \mp y\dot{y} = -x^2$ which integrates to

$$x^2 \mp y^2 = c - 2\int_0^t (x(s))^2\,\mathrm{d}s.$$

Thus $\lim_{t\to\infty}(x(t))^2 \mp (y(t))^2 = L \geqslant -\infty$. If $L = -\infty$, then $\lim_{t\to\infty} y(t) = \pm\infty$. Since $\lim_{t\to\infty} z(t)$ exists, then (x, y, z) all have limits at infinity. If $L > -\infty$, then either both $x(t)$ and $y(t)$ (and $z(t)$) have limits at infinity or any attractor for the system lies on a two-dimensional surface. Thus system (5.13) is non-chaotic.

System (5.14) can be rewritten as the system $\dot{x} = c\mathrm{e}^{-t}z + c^2\mathrm{e}^{-2t}$, $\dot{z} = x^2$. Thus $\lim_{t\to\infty} z(t) = L$. If $L \neq 0$, $x(t)$ is eventually monotone and has a limit at $t = \infty$. If $L = 0$, then $\lim_{t\to\infty} \dot{x}(t) = 0$ and any attractor for $(x(t), y(t), z(t))$ lies on a two-dimensional surface $x =$ constant. □

6. Four-term dissipative systems with constant terms

We now discuss four-term dissipative systems with constant terms. Clearly the equations already discussed would be simplified if any term were replaced by a constant and are thus still not chaotic. It is easily verified by inspecting the list of $4-1$ and $4-2$ systems that there are no non-trivial dissipative systems when any term is replaced by a constant. For $4-3$ systems there are the following non-trivial dissipative cases when one term is replaced by a constant:

$$\begin{cases} \dot{x} = \pm 1 + y^2 \\ \dot{y} = xz \\ \dot{z} = -z \end{cases} \tag{6.1}$$

$$\begin{cases} \dot{x} = 1 + yz \\ \dot{y} = x^2 \\ \dot{z} = -z \end{cases} \tag{6.2}$$

$$\begin{cases} \dot{x} = 1 - x \\ \dot{y} = xz \\ \dot{z} = y^2 \end{cases} \tag{6.3}$$

$$\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = xz \\ \dot{z} = 1 \end{cases} \tag{6.4}$$

$$\begin{cases} \dot{x} = yz - x \\ \dot{y} = x^2 \\ \dot{z} = 1. \end{cases} \tag{6.5}$$

Systems (6.1) and (6.2) have the familiar behaviour $z(t) = c\mathrm{e}^{-t} \to 0$ as $t \to \infty$ and thus any attractor lies in the plane $z = 0$.

For system (6.3) $x(t) \to 1$, $z(t) \uparrow L$ as $t \to \infty$ and thus both $x(t)$ and $z(t)$ are eventually of one sign. Thus $y(t)$ is monotone and has a limit as $t \to \infty$.

For system (6.4) $z(t) \to +\infty$ as $t \to \infty$ and so is eventually positive. $x(t)$ is either negative and increasing to a limit or eventually becomes positive. Thus $x(t)$ is also eventually of one sign. Thus $y(t)$ is monotone and has a limit L as $t \to \infty$. Thus $x(t) \to L$ as $t \to \infty$ and so (x, y, z) has a limit as $t \to \infty$.

For system (6.5) $z(t) \to +\infty$ as $t \to \infty$. Also $y(t) \uparrow L$ as $t \to \infty$. Thus $y(t)z(t) \to K$ as $t \to \infty$ and also $x(t) \to K$ as $t \to \infty$. Thus (x, y, z) has a limit at $t = \infty$.

7. Generalizations and conclusions

The arguments employed for four-term dissipative equations will carry over to most, but not all, non-dissipative equations. However, there are many more cases to consider. A partial analysis of this complexity has been carried out by Zhang [10]. Just the $4 - 1$ case alone has 810 different patterns. After eliminating equivalent systems (by permuting x, y, z) there are still 138 different types of $4 - 1$ equations. It turns out that 101 of these can be completely integrated, thereby eliminating the possibility of chaos. Of the remaining 37 patterns, 13 are essentially second-order autonomous systems and hence non-chaotic. Out of the remaining 24 cases eight are dissipative (analysed in section 3) and the remaining 16 are not dissipative. Many of these 16 cases are easily treated by our methods but not all. For example the system

$$\begin{cases} \dot{x} = xz + z \\ \dot{y} = x \\ \dot{z} = y \end{cases}$$

is equivalent to the scalar equation $\dddot{z} = z\ddot{z} + z$ which is difficult to analyse. Also, the system

$$\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = z \\ \dot{z} = x \end{cases}$$

is equivalent to the scalar equation $\dddot{y} = \ddot{y}^2 + y$, again hard to analyse.

For four-term equations with two nonlinearities there are 477 systems not permutationally equivalent. Of these, 134 are neither solvable, two-dimensional, nor reducible to linear systems. The 17 dissipative cases are treated in section 4. Two examples of non-dissipative cases that are hard to analyse are

$$\begin{cases} \dot{x} = xy + y^2 \\ \dot{y} = z \\ \dot{z} = x \end{cases}$$

which is equivalent to $\dddot{y} = y\ddot{y} + y^2$ and

$$\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = yz \\ \dot{z} = x \end{cases}$$

which is equivalent to $\dddot{z} = z\dot{z}\ddot{z} + z\ddot{z} - z\dot{z}^2$.

Not surprisingly four-term equations with three nonlinear terms can be even more complicated. For example

$$\begin{cases} \dot{x} = x^2 + xy \\ \dot{y} = yz \\ \dot{z} = x \end{cases}$$

reduces to the scalar equation $\dddot{y}y^2 = 3y\dot{y}\ddot{y} - 2\dot{y}^3 + y^2\ddot{y} - y\dot{y}^2$ and the system

$$\begin{cases} \dot{x} = xy + xz \\ \dot{y} = xy \\ \dot{z} = y \end{cases}$$

reduces to $\dddot{z}\dot{z} = \ddot{z}\dot{z}^2 + z\dot{z}\ddot{z} + \ddot{z}^2$.

Of course there are no four-term dissipative systems with four nonlinear terms. Without the assumption of dissipativity there seems little hope of analysing the many complicated systems which could arise.

It is very interesting that Sprott [8] has recently found an example of a five-term equation with only one nonlinearity which is both dissipative and chaotic. His example $\dddot{z} + A\ddot{z} - \dot{z}^2 + z = 0$ is the scalar form of

$$\begin{cases} \dot{x} = y^2 - z \\ \dot{y} = x - Ay \\ \dot{z} = y. \end{cases}$$

Many five-term equations with just one nonlinearity are amenable to the methods of this paper but clearly not all. It would be interesting to see how many distinct cases of dissipative $5 - 1$ chaos can exist.

We are also currently trying to extend this analysis to four-term conservative systems.

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