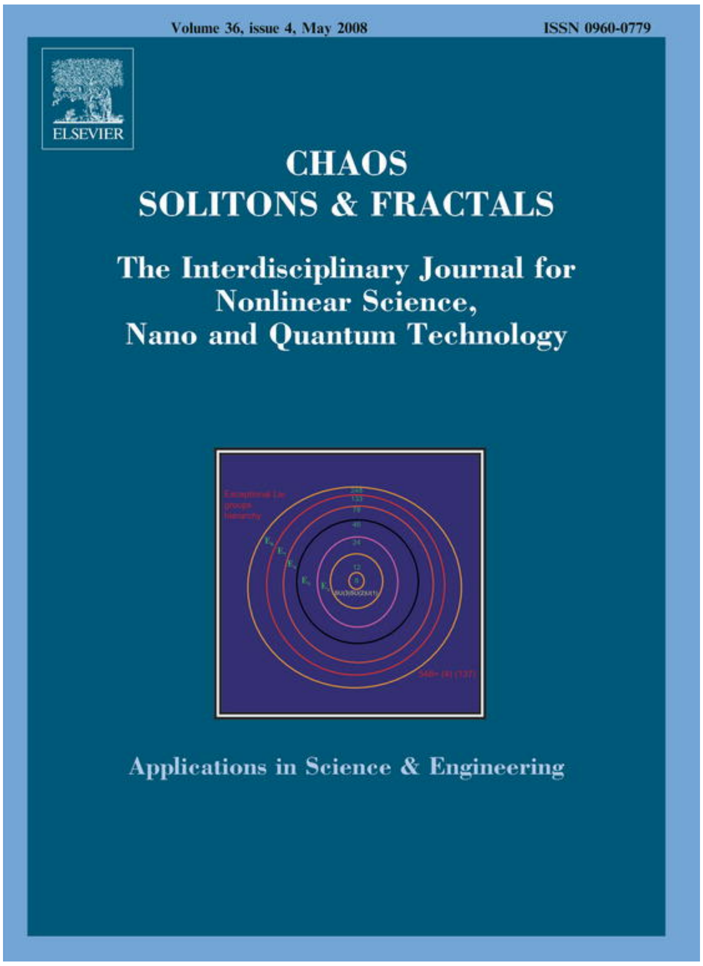
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Determining nonchaotic parameter regions in some simple chaotic jerk functions

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Abstract

In this paper we apply Theorem 2.1 in [Heidel J, Zhang F. Nonchaotic and chaotic behaviour in the three-dimensional quadratic systems: five-one conservative cases, in press] to some simple chaotic jerk functions listed in [Sprott JC. Simple chaotic systems and circuits. Am J Phys 2000;68(8):758–63; Sprott JC. Algebraically simple chaotic flows. Int J Chaos Theory Appl 2000;5(2):1–20] to locate the parameter regions at which they are nonchaotic. We show that for each of the twenty chaotic systems studied here there are some nonchaotic parameter regions. This indicates that our theorem will help reduce the amount of work searching for parameters causing chaos. We also generalize Theorem 2.1 to include systems with exponential functions.

1. Introduction

Recently progress in determining a system of ODEs being nonchaotic has been made [15]. The authors proved a general theorem, which provides sufficient conditions for solutions in 3D autonomous systems to be nonchaotic. Systematic studies of nonchaotic behavior of nonlinear ODEs trace back to work in 1997 [14] (see also [2]), in which was proposed a new idea for determining if a 3D autonomous system is nonchaotic. In [15] the idea was generalized to a theory. In this paper we apply the theorem to locate nonchaotic parameter and/or initial conditions regions in some chaotic systems. These systems include the Lorenz system [5], the Rössler equations [7] and some equations in jerk dynamics in the form $x''' = J(x'', x', x)$, where $J(x'', x', x)$ is called a jerk function and the equation is called a jerk equation, see [4,13].

Here we list the 20 chaotic autonomous systems and jerk functions we will study in this paper as follows:

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(A) Quadratic jerk equations

Lorenz equation

$$\begin{cases} x' = \sigma x + \sigma y \\ y' = -xz + rx - y \\ z' = xy - bz \end{cases} \tag{1.1}$$

where σ, r, and b are constants. The equations have chaos when $\sigma = 10$, $r = 28$, and $b = 8/3$.

Rössler equation

$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = b + xz - cz \end{cases} \tag{1.2}$$

where a, b, and c are constants. The system is chaotic when $a = b = 0.2$ and $c = 5.7$.

$$x''' + ax'' \pm x'^2 + x = 0 \tag{1.3}$$
$$x''' + ax'' \pm xx' + x = 0 \tag{1.4}$$
$$x''' + ax'' + x' = G(x) \tag{1.5}$$

where $G(x)$ is a second-degree (or higher) polynomial such as $x^2 - b$ or $x(x - b)$.

$$x''' + axx'' - x'^2 + x = 0 \tag{1.6}$$
$$x''' = -x'' - ax' - bx^2 + \frac{b}{4} \tag{1.7}$$
$$x''' = -x'' - ax' - bx + xx' \tag{1.8}$$
$$x''' = -ax'' + bx' + cx^2 + xx' - 1 \tag{1.9}$$
$$x''' = ax'' + bx' + cx^2 + xx'' - 1 \tag{1.10}$$
$$x''' = ax' + bx^2 + x'^2 - xx'' \tag{1.11}$$
$$x''' = ax'' + bx' + cx^2 + dx'^2 + exx' + xx'' - 1 \tag{1.12}$$
$$x''' = ax'' + bx' + x^2 - 1 \tag{1.13}$$
$$x''' = ax'' + bx + xx' - 1 \tag{1.14}$$

(B) Cubic jerk equations and other types

$$x''' + x'' + (T - R + Rx^2)x' + Tx = 0 \tag{1.15}$$
$$x''' = -ax' \pm (x - x^3) \tag{1.16}$$
$$x''' = -ax'' + bx' - x^3 - x \tag{1.17}$$
$$x''' + x'' + x^2x' + ax = 0 \tag{1.18}$$
$$x''' = -ax'' - e^{x'} - x \tag{1.19}$$
$$x''' = -ax'' - bx' + x \pm e^x \tag{1.20}$$

In systems (1.3)–(1.17), (1.19), (1.20), a, b, c, d, e, R, and T are constants.

Sprott [12] gave some parameter values at which the systems are chaotic. For the reader's convenience we list some of them here. (1.3) and (1.4), $a = 2.017$; (1.6), $a = 0.645$; (1.9), $a = 0.6$, $b = -3$, $c = 5$; (1.10), $a = -0.6$, $b = -2$, $c = 3$; (1.11), $a = 0.5$, $b = -1$; (1.12), $a = -1$, $b = 1$, $c = 2$, $d = -3$, $e = 1$; (1.13), $a = -0.5$, $b = -1.9$; (1.14), $a = -1.8$, $b = -2$; (1.16), $a = 3.7$; (1.17), $a = 0.6$, $b = 2.8$. The remaining systems (1.5), (1.7), (1.8), (1.15), (1.18), (1.19), and (1.20) can also be found in [12]. Also see [8,10,11].

Sprott [9] discovered that Eq. (1.3) is the simplest dissipative chaotic jerk function and showed that for $a = 2.017$, and initial conditions $(x, x', x'') = (0, 0, \mp 1)$, the Lyapunov exponents (base-e) are $(0.0550, 0, -2.0720)$ and Kaplan–Yorke dimension is $D_{KY} = 2.0265$. The range of a over which chaos occurs is quite narrow $(2.0168\ldots < a < 2.0577\ldots)$. Eq. (1.15) is known as the Moore–Spiegel oscillator and it is chaotic when $T = 6$ and $R = 20$, see [6,3].

We recall Theorem 1.2 in [15] in the following.

Consider the autonomous system

$$x' = f(x), \quad x \in \mathbb{R}^N, \ t \in \mathbb{R} \tag{1.21}$$

where $\prime = \frac{d}{dt}$, $f : \mathbb{R}^N \to \mathbb{R}^N$ is continuous. Let $x(0) = x_0$, and x_j, x_{0j} and f_j, $j = 1, 2, \ldots, N$ be the jth components of x, x_0 and f, respectively. Since we consider only bounded chaos in this paper, for convenience we call bounded chaos simply chaos.

Let $P(x) = \Sigma_\alpha A_\alpha x^\alpha$ be a polynomial, where $x \in \mathbb{R}^N$, $N \geqslant 1$ is an integer, $\alpha = (\alpha_1, \ldots, \alpha_N)$, and each of the α_i's is a non-negative integer, $x^\alpha = x_1^{\alpha_1} \ldots x_N^{\alpha_N}$, the order of the multi-index α is denoted by $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $A_\alpha \in \mathbb{R}$. By the fundamental theorem of algebra, every polynomial in the above form can be factored as the product of irreducible polynomials with real coefficients. Therefore for some $m \leqslant \max|\alpha|$

$$P(x) = \Pi_{j=1}^m P_j(x), \tag{1.22}$$

where $P_j(x)$'s are irreducible polynomials and the zero set $\{x \in \mathbb{R}^N | P(x) = 0\}$ of $P(x)$ is the union of the zero sets of $P_j(x)$, $j = 1, \ldots, m$. The zero set of each of the $P_j(x)$ consists of a finite number of connected components and each of them has dimension at most $N - 1$ in $\mathbb{R}^N$. We call each connected component of an irreducible polynomial a *simple surface*, a connected component that consists of more than one component of irreducible polynomials a *complex surface*. In this paper we also call a connected component an *isolated surface*.

For $N = 3$ an isolated surface can consist of a finite number of two-dimensional simple surfaces that are joined by a finite number of one-dimensional curves and/or points. From now on we consider the case $N = 3$ and make the following assumptions on system (1.21):

($H1$) System (1.21) is equivalent to

$$F^+(x_j'', x_j', x_j) = C^+ + \int_0^t G^+(x_j''(s), x_j'(s), x_j(s))\,ds, \quad ' = \frac{d}{dt} \tag{1.23}$$

for some integer $1 \leqslant j \leqslant 3$ and equivalent to

$$F^-(\ddot{x}_j, \dot{x}_j, x_j) = C^- + \int_0^\tau G^-(\ddot{x}_j(s), \dot{x}_j(s), x_j(s))\,ds, \quad \dot{} = \frac{d}{d\tau}, \ \tau = -t \tag{1.24}$$

for some integer $1 \leqslant j \leqslant 3$, where $C^\pm$ are constants and $G^\pm(y) \geqslant 0$, for all $y \in \mathbb{R}^3$.

($H2$) Each of the functions $f(y)$ in (1.21), $F^\pm(y)$, $G^\pm(y)$ in (2.5) and (1.28), $y \in \mathbb{R}^3$ is either a polynomial or a rational expression such that each of the simple surfaces is homeomorphic to a plane or a sphere or a subset of a plane or a sphere. Let

$$G^\pm(y) = \frac{R_0^\pm(y)}{Q_0^\pm(y)}, \quad F^\pm(y) = \frac{R^\pm(y)}{Q^\pm(y)} \quad \text{and} \quad f(y) = \frac{r(y)}{q(y)}$$

where $R^\pm(y)$, $Q^\pm(y)$, $r(y)$ and $q(y)$ are polynomials. We assume that $Q_0^\pm(y) > q_0 > 0$, $|Q^\pm(y)| > Q_1 > 0$, and $|q^\pm(y)| > q_1 > 0$ for some positive constants q_0, Q_1, and q_1.

Theorem 1.1 [15]. *Let $N = 3$. Under hypotheses ($H1$) and ($H2$) system (1.21) has no bounded chaos.*

Next we extend theorem 1.1 in the following way. Let f be in the following form:

$$P_e(x) = \Sigma_\alpha e^{\Sigma_\beta B_\beta x^\beta} A_\alpha x^\alpha. \tag{1.25}$$

for some $m \leqslant \max|\alpha|$ and $k \leqslant \max|\beta|$ and

$$P_e(x) = \Pi_{j=1}^m P_{ej}(x), \tag{1.26}$$

where the $P_{ej}(x)$'s are in the form (1.25) and irreducible. The zero set $\{x \in \mathbb{R}^N | P_e(x) = 0\}$ of $P_e(x)$ is the union of the zero sets of $P_{ej}(x)$, $j = 1, \ldots, m$. The zero set of each of the $P_{ej}(x)$ consists of a finite number of connected components and each of them has dimension at most $N - 1$ in $\mathbb{R}^N$. Here we consider the case $N = 3$.

We make the following assumptions on system (1.21):

($A1$) System (1.21) is equivalent to

$$F^+(x_j'', x_j', x_j, e^{\Lambda_1^+}, \ldots, e^{\Lambda_{n^+}^+}) = C^+ + \int_0^t G^+(x_j''(s), x_j'(s), x_j(s), e^{\Theta_1^+}, \ldots, e^{\Theta_{m^+}^+})\,ds, \tag{1.27}$$

for some integer $1 \leqslant j \leqslant 3$, where $\prime = \frac{d}{dt}$, and equivalent to

$$F^-(\ddot{x}_j, \dot{x}_j, x_j, e^{\Lambda_1^-}, \ldots, e^{\Lambda_{n^-}^-}) = C^- + \int_0^\tau G^-(\ddot{x}_j(s), \dot{x}_j(s), x_j(s), e^{\Theta_1^-}, \ldots, e^{\Theta_{m^-}^-})\,ds, \tag{1.28}$$

for some integer $1 \leqslant j \leqslant 3$, *where* $\dot{} = \frac{d}{d\tau}$, $\tau = -t$, $C^{\pm}$ *are constants,* $\Lambda^{\pm}_{k^{\pm}}$, $\Theta^{\pm}_{i^{\pm}}$ *are polynomials in* (x''_j, x'_j, x_j) *and* $G^{\pm}(y) \geqslant 0$ *for all* $y \in \mathbb{R}^{3+m^{\pm}}$.

(*A*2) *Each of the functions* $f(y)$ *in* (1.21), $y \in \mathbb{R}^3$, $F^{\pm}(y)$, $y \in \mathbb{R}^{3+n^{\pm}}$, $G^{\pm}(y)$, $y \in \mathbb{R}^{3+m^{\pm}}$ *and* $\Lambda^{\pm}_{k^{\pm}}(y)$ *and* $\Theta^{\pm}_{i^{\pm}}(y)$, $y \in \mathbb{R}^3$ *in* (1.27) *and* (1.28), *is either a polynomial or a rational expression such that each of the simple surfaces is homeomorphic to a plane or a sphere or a subset of a plane or a sphere. Let*

$$G^{\pm}(y) = \frac{R_0^{\pm}(y)}{Q_0^{\pm}(y)}, \quad F^{\pm}(y) = \frac{R^{\pm}(y)}{Q^{\pm}(y)} \quad \text{and} \quad f(y) = \frac{r(y)}{q(y)}$$

where $R^{\pm}(y)$, $Q^{\pm}(y)$, $r(y)$ *and* $q\ (y)$ *are polynomials. We assume that* $Q_0^{\pm}(y) > q_0 > 0$, $|Q^{\pm}(y)| > Q_1 > 0$, *and* $|q^{\pm}(y)| > q_1 > 0$ *for some positive constants* q_0, Q_1, *and* q_1.

Theorem 1.2. *Let* $N = 3$. *Under hypotheses* (*A*1) *and* (*A*2) *system* (1.21) *has no bounded chaos.*

The proof is the same as that of Theorem 1.1 in [15].

2. Quadratic jerk equations

First we determine nonchaotic parameter regions of the Lorenz equations and Rössler equation.

Consider the Lorenz equations

$$\begin{cases} x' = \sigma x + \sigma y \\ y' = -xz + rx - y \\ z' = xy - bz \end{cases} \tag{2.1}$$

where σ, r, and b are constants.

Theorem 2.1. *If* (1) $\sigma \geqslant 0$, $0 \leqslant b \leqslant 2\sigma + 2$, $r \leqslant 1$ *or* $\sigma \geqslant 0$, $b \leqslant 0$, $r \geqslant 1$.

(2) $\sigma \leqslant 0$, $b \geqslant max\{2\sigma + 2, 0\}$, $r \leqslant 1$ *or* $\sigma \leqslant -1$; *or* $2\sigma + 2 \leqslant b \leqslant 0$, $r \geqslant 1$, *then the Lorenz system* (2.1) *is not chaotic.*

Proof. From the first equation of (2.1), we have

$$x'' = -\sigma x' + \sigma y' = -\sigma x' + \sigma(-xz + rx - y)$$

$$x'' = -\sigma x' + \sigma y' = -\sigma x' + \sigma\left(-xz + rx - \frac{x'}{\sigma} - x\right)$$

$$x''' = -\sigma x'' + \sigma\left(-x'z - xz' - \frac{x''}{\sigma} - x' + rx'\right)$$

$$x''' = -\sigma x'' - \sigma x'z + \sigma x(xy - bz) - x'' - \sigma x' + \sigma r x'$$

Multiply both sides of the equation by x. We have

$$xx''' = -\sigma xx'' - \sigma x'xz + \sigma x^3\left(\frac{1}{\sigma}x' + x\right) - b\sigma x^2 z - xx'' - \sigma xx' + \sigma r xx'$$

Since $\sigma xz = -x'' - \sigma x' + \sigma\, rx - x' - \sigma x$

$$xx''' = -\sigma xx'' - x'(-x'' - \sigma x' + \sigma rx - x' - \sigma x) + x^3 x' + \sigma x^4 - bx(-x'' - \sigma x' + \sigma rx - x' - \sigma x) - xx'' - \sigma xx' + \sigma r xx'$$

Integrating this equation, we obtain

$$xx'' - \frac{1}{2}x'^2 - \frac{1}{2}b(\sigma + 1)x^2 - \frac{1}{2}x'^2 - (b - \sigma - 1)xx' - \frac{1}{4}x^4 = C + \int_0^t (\sigma x^4 - (b - 2\sigma - 2)x'^2 + b\sigma(1 - r)x^2)\,ds$$

where C is a constant and $t \geqslant 0$. We need to calculate the following inequalities:

$$\begin{cases} \sigma \geqslant 0 \\ b - 2\sigma - 2 \leqslant 0 \\ b\sigma(1 - r) \geqslant 0 \end{cases} \quad \text{or} \quad \begin{cases} \sigma \leqslant 0 \\ b - 2\sigma - 2 \geqslant 0 \\ b\sigma(1 - r) \leqslant 0 \end{cases}$$

For the first set of inequalities, we obtain condition (1)

$$\begin{cases} \sigma \geqslant 0 \\ 0 \leqslant b \leqslant 2\sigma + 2 \\ r \leqslant 1 \end{cases} \quad \text{or} \quad \begin{cases} \sigma \geqslant 0 \\ b \leqslant 0 \\ r \geqslant 1 \end{cases}$$

For the second set, we obtain condition (2)

$$\begin{cases} \sigma \leqslant 0 \\ b \geqslant \max\{2\sigma + 2, 0\} \\ r \leqslant 1 \end{cases} \quad \text{or} \quad \begin{cases} \sigma \leqslant -1 \\ 2\sigma + 2 \leqslant b \leqslant 0 \\ r \geqslant 1 \end{cases}$$

When condition (1) or (2) is satisfied, $\sigma x^4 - (b - 2\sigma - 2)x'^2 + b\sigma(1 - r)x^2 \geqslant 0$ or $\leqslant 0$. By Theorem 1.1, the Lorenz system is not chaotic under condition (1) or (2). $\square$

Note that Lorenz equation has chaos when $\sigma = 10$, $r = 28$, and $b = 8/3$. An interesting condition ($\sigma \geqslant 0$, $0 \leqslant b \leqslant 2\sigma + 2$ and $r \leqslant 1$) obtained by Theorem 2.1 shows a parameter region where the Lorenz equations do not exhibit chaos.

Consider the Rössler equations

$$\begin{cases} x' = -y - z \\ y' = x + ay \\ z' = b + xz - cz \end{cases} \tag{2.2}$$

where a, b, and c are constants.

Theorem 2.2. *If* $(c + 1)^2 - 4ab \leqslant 0$, *the Rössler system* (2.2) *is not chaotic.*

Proof. From the second equation of (2.2), we have

$$y''' - ay'' = x'' = -y' - z'$$
$$y''' - ay'' = -y - b - xz + cz$$

Since $x = y' - ay$, we have $z = -y - x' = -y - y'' + ay'$. Thus

$$y''' - ay'' = -y - b - (y' - ay)(-y - y'' + ay') + c(-y - y'' + ay')$$
$$y''' - ay'' = -y - b + yy' + y'y'' - ay'^2 - ay^2 - ayy'' + a^2yy' - cy - cy'' + acy'$$

thus

$$y''' - ay'' = (-1 - c)y - b + (1 + a^2)yy' + y'y'' - ay^2 - a(yy')' - cy'' + acy'$$
$$y'' - ay' = \frac{1}{2}(1 + a^2)y^2 + \frac{1}{2}y'^2 - ayy' - cy' + acy - \int_0^t (ay^2 + (1 + c)y + b)\,\mathrm{d}s + A$$

where A is a constant and $t \geqslant 0$. When condition $(c + 1)^2 - 4ab \leqslant 0$ is satisfied, $ay^2 + (1 + c)y + b \geqslant 0$ or $\leqslant 0$. By Theorem 1.1 the Rössler system is not chaotic under the given conditions.

Next we are going to determine nonchaotic parameter regions of some simple chaotic jerk functions which were discovered by Sprott and others. $\square$

Consider equation

$$x''' + ax'' \pm x'^2 + x = 0 \tag{2.3}$$

Theorem 2.3. *If* $a < 0$, *Eq.* (2.3) *is not chaotic.*

Proof. Multiply both sides of Eq. (2.3) by x''. We have

$$x''x''' + ax''^2 \pm x'^2x'' + xx'' = 0$$

Integrate the equation to get

$$\frac{1}{2}x''^2 \pm \frac{1}{3}x'^3 + xx' = C + \int_0^t (x'^2 - ax''^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $x'^2 - ax''^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. $\square$

Consider equation

$$x''' + ax'' \pm xx' + x = 0 \tag{2.4}$$

Theorem 2.4. *If $a < 0$, Eq. (2.4) is not chaotic.*

Proof. Multiply both sides of Eq. (2.4) by x.
We have

$$xx''' + axx'' \pm x^2x' + x^2 = 0$$

Integrate the equation to get

$$xx''^2 - \frac{1}{2}x'^2 + axx' \pm \frac{1}{3}x^3 = C + \int_0^t (ax'^2 - x^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $ax'^2 - x^2 \leqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider the equation

$$x''' + ax'' + x' = G(x) \tag{2.5}$$

where $G(x)$ is a second-degree (or higher) polynomial such as $x^2 - b$ or $x(x - b)$. a and b are constants. We have the following theorem.

Theorem 2.5. *If $G(x) > 0$, $aG_x(x) \geqslant 0$, or $axG(x) \geqslant 0$ and $a \neq 0$, Eq. (2.5) is not chaotic.*

Proof. If $G(x) > 0$ integrate Eq. (2.5) to get

$$x'' + ax' + x = \int_0^t G(x)\,\mathrm{d}s + C$$

where C is a constant and $t \geqslant 0$. By theorem, this equation is not chaotic.
If $aG_x(x) \geqslant 0$, multiply both sides of Eq. (2.5) by x''. We have

$$x''x''' + ax''^2 + x'x'' = G(x)x''$$

Integrate the equation to get

$$\frac{1}{2}x''^2 + \frac{1}{2}x'^2 - G(x)x' = C - \int_0^t (ax''^2 + x'^2G_x(x))\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. Since $aG_x(x) \geqslant 0$, $ax''^2 + x'^2G_x(x) \geqslant 0$ or $\leqslant 0$. By the theorem, this equation is not chaotic.
If $axG(x) \geqslant 0$, and $a \neq 0$, multiply both sides of Eq. (2.5) by x. We have

$$xx''' + axx'' + xx' = G(x)x$$

Integrate the equation to get

$$xx'' - \frac{1}{2}x'^2 + axx' + \frac{1}{2}x^2 = C + \int_0^t (ax'^2 + xG(x))\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. Since $axG(x) \geqslant 0$, $ax'^2 + xG(x) \geqslant 0$ or $\leqslant 0$. By Theorem 1.1, this equation is not chaotic. This completes the proof of the theorem. □

For example: Let $G_1(x) = x^2 - b$ and $G_2(x) = x^3 - b$.
If we take $b < 0$, $G_1(x) = x^2 - b \geqslant 0$, then this equation is not chaotic.
If $a > 0$ and $G_{2x}(x) = 3x^2$, then $ax'^2 + G_x(x)x'^2 = ax''^2 + 3x^2x'^2 \geqslant 0$. So this equation is not chaotic either.
Note that our theorem cannot deal with the case $G(x) = x(x - b)$.
Consider the following equation:

$$x''' + axx'' - x'^2 + x = 0 \tag{2.6}$$

Theorem 2.6. (1) *If $a = -2$, or*
(2) *If $a < 0$ and $x''(0) \geqslant -\frac{1}{a}$, $x(0)$, $x'(0) \in \mathbb{R}$, Eq. (2.6) is not chaotic.*

Proof. (1) Multiply both sides of Eq. (2.6) by x'. We have

$$x'x''' + axx'x'' - x'^3 + xx' = 0$$

Integrate the equation to get

$$xx'' - \int_0^t x''^2\,\mathrm{d}s + \int_0^t axx'x''\,\mathrm{d}s - \int_0^t x'^3\,\mathrm{d}s + \frac{1}{2}x^2 = C$$

Since

$$\int_0^t xx'x''\,\mathrm{d}s = \frac{1}{2}xx'^2 - \frac{1}{2}\int_0^t x'^3\,\mathrm{d}s$$

we have

$$xx'' + \frac{a}{2}xx'^2 + \frac{1}{2}x^2 = C + \int_0^t x''^2\,\mathrm{d}s + \left(\frac{a}{2}+1\right)\int_0^t x'^3\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a = -2$, by Theorem 1.1, this equation is not chaotic.

(2) Multiply both sides of Eq. (2.6) by integration factor $\mathrm{e}^{a\int_0^t x\,\mathrm{d}s}$. We have

$$\mathrm{e}^{a\int_0^t x\,\mathrm{d}s}x''' + ax\mathrm{e}^{a\int_0^t x\,\mathrm{d}s}x'' - x'^2\mathrm{e}^{a\int_0^t x\,\mathrm{d}s} + x\mathrm{e}^{a\int_0^t x\,\mathrm{d}s} = 0$$

and so

$$\left(\mathrm{e}^{a\int_0^t x\,\mathrm{d}s}x'' + \frac{1}{a}\mathrm{e}^{a\int_0^t x\,\mathrm{d}s}\right)' = x'^2\mathrm{e}^{a\int_0^t x\,\mathrm{d}s}$$

Integrate the equation to get

$$x''(t) = -\frac{1}{a} + \left(x''(0) + \frac{1}{a} + \int_0^t x'^2\mathrm{e}^{a\int_0^s x(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right)\mathrm{e}^{-a\int_0^t x\,\mathrm{d}s}$$

where C is a constant and $t \geqslant 0$. Since $a < 0$ and $x''(0) \geqslant -\frac{1}{a}$, $x(0)$, $x'(0) \in \mathbb{R}$, so $x''(t) > 0$. The equation is not chaotic. □

Consider the equation

$$x''' = -x'' - ax' - bx^2 + \frac{b}{4} \tag{2.7}$$

Theorem 2.7. *If $a < 0$ and $b \in \mathbb{R}$, Eq. (2.7) is not chaotic.*

Proof. Multiply both sides of Eq. (2.7) by x'. We have

$$x'x''' = -x'x'' - ax'^2 - bx^2x' + \frac{b}{4}x'$$

Integrate the equation to get

$$x'x'' + \frac{1}{2}x'^2 + \frac{b}{3}x'^3 - \frac{b}{4}x = C + \int_0^t (x''^2 - ax'^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $x''^2 - ax'^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = -x'' - ax' - bx + xx' \tag{2.8}$$

Theorem 2.8. *If $b < 0$ and $a \in \mathbb{R}$, Eq. (2.8) is not chaotic.*

Proof. Multiply both sides of Eq. (2.8) by x. We have

$$xx''' = -xx'' - axx' - bx^2 + x^2x'$$

Integrate the equation to get

$$xx'' - \frac{1}{2}x'^2 + \frac{a}{2}x^2 + xx' - \frac{1}{3}x^3 = C + \int_0^t (x'^2 - bx^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $b < 0$, $x'^2 - bx^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = -ax'' + bx' + cx^2 + xx' - 1 \tag{2.9}$$

Theorem 2.9. *If $c < 0$ or $c = 0.5$ and $a, b \in \mathbb{R}$, Eq. (2.9) is not chaotic.*

Proof. Integrate Eq. (2.9) to get

$$x'' + ax' - bx - \frac{1}{2}x^2 = C + \int_0^t (cx^2 - 1)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $c < 0$, $a, b \in \mathbb{R}$, $cx^2 - 1 < 0$. By the theorem, this equation is not chaotic.

Multiply both sides of Eq. (2.9) by x'' and integrate to get

$$\frac{1}{2}x''^2 - \frac{b}{2}x'^2 - cx^2x' + x' = C - \int_0^t x''^2\,\mathrm{d}s + (1-2c)\int_0^t xx'x''\,\mathrm{d}s$$

When $c = 0.5$ and $a, b \in \mathbb{R}$, by Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = ax'' + bx' + cx^2 + xx'' - 1 \tag{2.10}$$

Theorem 2.10. *If $c < 0$ and $a, b \in \mathbb{R}$, Eq. (2.10) is not chaotic.*

Proof. Integrating Eq. (2.10), we have

$$x'' - ax' - bx - xx' = C + \int_0^t (cx^2 - x'^2 - 1)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $c < 0$, $a, b \in \mathbb{R}$, $cx^2 - x'^2 - 1 < 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = ax' + bx^2 + x'^2 - xx'' \tag{2.11}$$

Theorem 2.11. *If $b > 0$ and $a \in \mathbb{R}$, Eq. (2.11) is not chaotic.*

Proof. Integrate Eq. (2.11) to get

$$x'' - ax + xx' = C + \int_0^t (bx^2 + 2x'^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $b > 0$, $a \in \mathbb{R}$, $bx^2 + 2x'^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider the equation

$$x''' = ax'' + bx' + cx^2 + dx'^2 + exx' + xx'' - 1 \tag{2.12}$$

Each of $e = 0$ or $e \neq 0$ present a different chaotic jerk function.

Theorem 2.12. *If (1) $c \leqslant 0$, $d \leqslant 0$, and $a, b \in \mathbb{R}$; or (2) $c \leqslant 0$, $d = 1$ and $a, b, e \in \mathbb{R}$, Eq. (2.12) is not chaotic.*

Proof. Integrate Eq. (2.12) to get

$$x'' - ax' - bx - xx' - \frac{e}{2}x^2 = C + \int_0^t (cx^2 + dx'^2 - x'^2 - 1)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$.

(1) When $c \leqslant 0$, $d \leqslant 0$, or (2) When $c \leqslant 0$, $d = 1$, $a, b, e \in \mathbb{R}$, $cx^2 + dx'^2 - x'^2 - 1 < 0$. By Theorem 1.1 the equation is not chaotic. □

Consider equation

$$x''' = ax'' + bx' + x^2 - 1 \tag{2.13}$$

Theorem 2.13. *If $b > 0$, Eq. (2.13) is not chaotic.*

Proof. Multiply both sides of Eq. (2.13) by x'. We have

$$x'x''' = ax'x'' + bx'^2 + (x^2 - 1)x'$$

Integrate the equation to get

$$x'x'' - \frac{a}{2}x'^2 + \frac{1}{3}x^3 + x = C + \int_0^t (x''^2 + bx'^2)\,ds$$

where C is a constant and $t \geqslant 0$. When $b > 0$, $x''^2 + bx'^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = ax'' + bx + xx' - 1 \tag{2.14}$$

Since we can not deal with the jerk function using the main theorem in [15], we are going to study the function by another method. Let $P_0 = (x(0), y(0), z(0))$ be initial values.

Theorem 2.14. *If $b > 0$, $a > 0$ and $x(0) > \frac{1}{b}$, $y(0)$, $z(0) > 0$, then Eq. (2.14) is not chaotic.*

Proof. We rewrite Eq. (2.14) as a system

$$\begin{cases} x' = y \\ y' = z \\ z' = az + bx + xy - 1 \end{cases}$$

and note that

$$-xx' = -xy, \quad -ay' = -az$$

Adding the last three equations, we have

$$\left(z - ay - \frac{1}{2}x^2\right)' = bx - 1$$

When $b > 0$, $a > 0$ and we choose $x(0) > \frac{1}{b}$, $y(0) > 0$, $z(0) > 0$, then

$$x'(0) > 0, \quad y'(0) > 0, \quad z'(0) = az(0) + bx(0) + x(0)y(0) - 1 > 0, \quad \text{and} \quad \left(z - ay - \frac{1}{2}x^2\right)'(0) = bx(0) - 1 > 0.$$

Therefore there exists a $\eta > 0$ such that when $t \in (0, \eta)$, $\left(z - ay - \frac{1}{2}x^2\right)' = bx - 1 > 0$, $x' > 0$, $y' > 0$, and $z' = az + bx + xy - 1 > 0$.

Suppose that there is a finite $T > \eta$ such that $x(T) = \frac{1}{b}$. This implies that there exists a $t^* \in (\eta, T)$ such that $x'(t^*) = y(t^*) = 0$. Thus there exists a $t^{**} \in (\eta, t^*]$ such that $y'(t^{**}) = z(t^{**}) = 0$. Since $\left(z - ay - \frac{1}{2}x^2\right)' = bx - 1 > 0$, for $t \in (\eta, t^{**}]$, $z - ay - \frac{1}{2}x^2 \nearrow$ and $-ay - \frac{1}{2}x^2 \searrow$. Hence $z \nearrow$ for $t \in (\eta, t^{**}]$. This contradicts $z(t^{**}) = 0$. Thus $\left(z - ay - \frac{1}{2}x^2\right)' = bx - 1 > 0$, $x' > 0$, $y' > 0$ and $z' > 0$, for all $t > 0$. Hence this equation is not chaotic under the given conditions. □

3. Cubic jerk equations and other types

Consider equation

$$x''' + x'' + (T - R + Rx^2)x' + Tx = 0. \tag{3.1}$$

Theorem 3.1. *If $T < 0$ and $R \in \mathbb{R}$, Eq. (3.1) is not chaotic.*

Proof. Multiply both sides of Eq. (3.1) by x. We have

$$xx''' + xx'' + (T - R + x^2)xx' + Tx^2 = 0.$$

Integrating the equation to get

$$xx'' - \frac{1}{2}x'^2 + xx' + \frac{T-R}{2}x^2 + \frac{R}{4}x^4 = C + \int_0^t (x'^2 - Tx^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $T < 0$, $x'^2 - Tx^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = -ax' \pm (x - x^3) \tag{3.2}$$

Theorem 3.2. *If $a < 0$, Eq. (3.2) is not chaotic.*

Proof. Multiply both sides of Eq. (3.2) by x'. We have

$$x'x''' = -ax'^2 \pm (x - x^3)x'$$

Integrate the equation to get

$$x'x'' \mp \left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right) = C + \int_0^t (x''^2 - ax'^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $x''^2 - ax'^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' = -ax'' + bx' - x^3 - x \tag{3.3}$$

Theorem 3.3. *If $a < 0$, Eq. (3.3) is not chaotic.*

Proof. Multiply both sides of Eq. (3.3) by x''. We have

$$x''x''' = -ax''^2 + bx'x'' - x'^3x'' - xx''$$

Integrate the equation to get

$$\frac{1}{2}x''^2 - \frac{b}{2}x'^2 + \frac{1}{4}x'^4 + xx' = C + \int_0^t (x'^2 - ax''^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $x'^2 - ax''^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

Consider equation

$$x''' + x'' + x^2x' + ax = 0 \tag{3.4}$$

Theorem 3.4. *If $a < 0$, Eq. (3.4) is not chaotic.*

Proof. Eq. (3.4) can be written as

$$\begin{cases} x' = y \\ y' = z \\ z' = -z - x^2y - ax \end{cases}$$

which leads to

$$\begin{aligned} &xz' = -xz - x^3y - ax^2 \\ &zx' = zy \\ &-yy' = -zy \\ &x^3x' = x^3y \\ &xy' = xz \\ &yx' = y^2 \end{aligned}$$

Add the above six equations and integrate to get

$$xz + xy - \frac{1}{2}y^2 + \frac{1}{4}x^4 = C + \int_0^t (y^2 - ax^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $y^2 - ax^2 \geqslant 0$. By Theorem 1.1 this equation is not chaotic. □

The following jerk functions are of exponential types

Consider equation

$$x''' = -ax'' - e^{x'} - x \tag{3.5}$$

Theorem 3.5. *If $a < 0$, Eq. (3.5) is not chaotic.*

Proof. Multiply both sides of Eq. (3.5) by x''. We have

$$x''x''' = -ax''^2 - e^{x'}x'' - xx''$$

Integrate the equation to get

$$\frac{1}{2}x''^2 + e^{x'} + xx' = C + \int_0^t (x'^2 - ax''^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $a < 0$, $x'2 - ax''^2 \geqslant 0$. By Theorem 1.2 this equation is not chaotic. □

Consider equation

$$x''' = -ax'' - bx' + x \pm e^x \tag{3.6}$$

Theorem 3.6. *If $b < 0$, Eq. (3.6) is not chaotic.*

Proof. Multiply both sides of Eq. (3.6) by x'. We have

$$x'x''' = -ax'x'' - bx'^2 + xx' \pm e^x x'$$

Integrate the equation to get

$$x'x'' + \frac{a}{2}x'^2 - \frac{1}{2}x^2 \mp e^x = C + \int_0^t (x''^2 - bx'^2)\,\mathrm{d}s$$

where C is a constant and $t \geqslant 0$. When $b < 0$, $x''^2 - bx'^2 \geqslant 0$. Again by Theorem 1.2 this equation is not chaotic. □

4. Conclusion

Our study in this paper indicates that our theorem in [15] can be applied to a large number of chaotic ODEs and jerk functions to determine nonchaotic parameters. However we can not provide a clear cut between chaotic and nonchaotic parameter regions. This is of course very difficult. There are also coupled chaotic jerk dynamical systems, such as

$$x''' + ax'' + x' = x^2 - bx$$
$$x''' + ax'' - xx'^2 + x^3 = 0$$

for which our theory can not produce useful result.

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