Problem of the week #7: Solutions

Solution 1. Define \( f(T) = T^3 + aT^2 + bT + c \). By the fundamental theorem of algebra, it can be factored as \( f(T) = (T - \alpha)(T - \beta)(T - \gamma) \) for three (not necessarily distinct) roots \( \alpha, \beta, \gamma \). Expanding yields:

\[
f(T) = T^3 - (\alpha + \beta + \gamma)T^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)T - (\alpha\beta\gamma).
\]

Vieta’s formulas state that, for monic (i.e. leading coefficient 1) polynomials of any degree, each coefficient is equal to \( \pm \) a corresponding elementary symmetric polynomial of the roots \( \alpha, \beta, \gamma \).

In this case, we have:

\[
\begin{align*}
\alpha + \beta + \gamma &= -a \\
\alpha\beta + \beta\gamma + \gamma\alpha &= b \\
\alpha\beta\gamma &= -c
\end{align*}
\]

On the other hand, define \( g(T) = (T - \alpha^2)(T - \beta^2)(T - \gamma^2) \), and assume it expands as \( g(T) = T^3 + AT^2 + BT + C \), then Vieta’s formulas say

\[
\begin{align*}
\alpha^2 + \beta^2 + \gamma^2 &= -A \\
(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 &= B \\
(\alpha\beta\gamma)^2 &= -C
\end{align*}
\]

The easiest to find is \( C = -(\alpha\beta\gamma)^2 = -c^2 \).

Next, notice \( (-a)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) \) (after regrouping and combining like terms) which is \( -A + 2b \), and so \( A = 2b - a^2 \).

Finally, \( b^2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2) \), by the same token. The latter part may be factored as \( 2\alpha\beta\gamma(\alpha + \beta + \gamma) \), so this equation states \( b^2 = B + 2(-c)(-a) \), and thus \( B = b^2 - 2ac \).

Putting it all together, we conclude

\[
g(T) = T^3 + (2b - a^2)T^2 + (b^2 - 2ac)T - c^2.
\]
Solution 2. The formula $A^2 - B^2 = (A - B)(A + B)$, which says a difference of squares factors as a product of conjugates, may be used:

$$g(T) = (T - \alpha^2)(T - \beta^2)(T - \gamma^2)$$

$$= (\sqrt{T} - \alpha)(\sqrt{T} + \alpha) \cdot (\sqrt{T} - \beta)(\sqrt{T} + \beta) \cdot (\sqrt{T} - \gamma)(\sqrt{T} + \gamma)$$

$$= (\sqrt{T} - \alpha)(\sqrt{T} - \beta)(\sqrt{T} - \gamma) \cdot (\sqrt{T} + \alpha)(\sqrt{T} + \beta)(\sqrt{T} + \gamma),$$

valid for $T \geq 0$, or even for $T < 0$ if we adopt the convention $\sqrt{-x} = ix$ whenever $-x$ is negative. The first three factors are $f(\sqrt{T})$, however the last three terms have $+$ signs. To remedy this, multiply by $(-1)^4$ and distribute the $(-1)$s out like so:

$$(\sqrt{T} + \alpha)(\sqrt{T} + \beta)(\sqrt{T} + \gamma) = -(-\sqrt{T} - \alpha)(-\sqrt{T} - \beta)(-\sqrt{T} - \gamma).$$

Thus, we have $g(T) = -f(\sqrt{T})f(-\sqrt{T})$. Multiplying this out,

$$g(T) = (T^{3/2} + aT + bT^{1/2} + c)(T^{3/2} - aT + bT^{1/2} - c)$$

$$= T^3 + (2b - a^2)T^2 + (b^2 - 2ac)T - c^2$$

The fractional powers cancel out in the end. (Interpret $T^{3/2}$ and $T^{1/2}$ as placeholders for $T\sqrt{T}$ and $\sqrt{T}$ for negative numbers if necessary.)