Washers in Balance: Solution

Say the slice volumes are $V_1, \cdots, V_n$ and the radius is r (all depending on n).

Let A_i be the area of a cross-section of the corresponding slice of a unit sphere. Scaling by r^2, we can say $r^2 A_i$ is the area of a cross-section of the volume V_i. Its height is $\frac{2r}{n}$, so we may approximate $V_i \approx (\frac{2r}{n})(r^2 A_i)$.

The volume equation $\frac{4}{3}\pi r^3 = Cn$ implies $\frac{2r^3}{n} = \frac{3C}{2\pi}$ which means $V_i \approx (\frac{3C}{2\pi})A_i$.

The logarithm $\ln P_n = \ln V_1 + \cdots + \ln V_n$ we may estimate by

$$\begin{aligned} &\approx \quad n\ln\left(\tfrac{3C}{2\pi}\right) + \ln A_1 + \cdots + \ln A_n \\ &= \quad n\left[\ln\left(\frac{3C}{2\pi}\right) + \frac{1}{n}\sum_{i=1}^{n} \ln A_i\right] \end{aligned}$$

For the limit to exist, the expression inside brackets must tend to 0 as $n \to \infty$.

The summation is akin to a Riemann sum. But for what integral? We need to parametrize the cross-sectional areas using the interval $[0, 1]$. The cross-sections are circles with radii $\sqrt{1-z^2}$ and areas $A = \pi(\sqrt{1-z^2})^2$, and we can parametrize $-1 \le z \le 1$ from $0 \le t \le 1$ using $z(t) = 2t - 1$. Thus, the cross-sectional area is $A(t) = \pi\big(1 - (2t-1)^2\big) = 4\pi t(1-t)$, and then

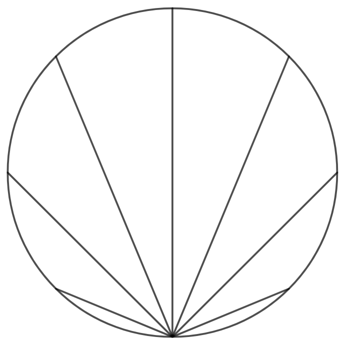
$$\begin{aligned} \ln\left(\frac{3C}{2\pi}\right) &= -\int_0^1 \ln\big[4\pi t(1-t)\big]\,\mathrm{d}t. \\ &= -\ln(4\pi) - \int_0^1 \ln t\,\mathrm{d}t - \int_0^1 \ln(1-t)\,\mathrm{d}t \end{aligned}$$

We may add $\ln(4\pi)$ over to the left, then combine the integrals on the right since they are equal (by symmetry - use the substitution $s = 1 - t$):

$$\ln(6C) = -2\int_0^1 \ln t\,\mathrm{d}t = -2\Big[t(\ln t - 1)\Big]_0^1 = 2$$

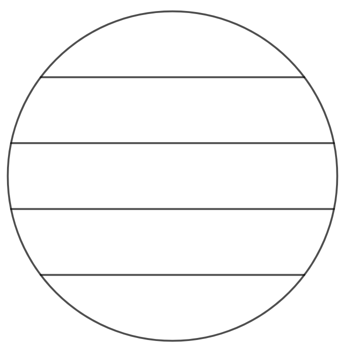
(since $\lim\limits_{t\to 0^+} t\ln t = 0$), which leads to $C = \dfrac{e^2}{6}$.

Similar results are possible slicing up a circle by chords (equal angle apart), into parallel chords (equal distance apart), or a sphere into concentric shells (of equal thickness), provided by Dan (user 398708) on Math.StackExchange:



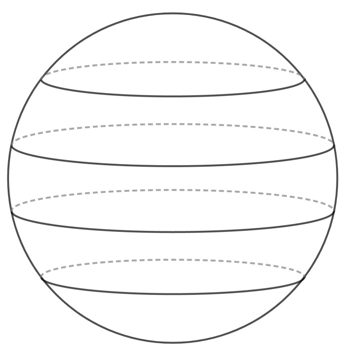
$$A = 2n$$

$$P \to [2\cosh(\tfrac{\pi}{2\sqrt{3}})]^2$$



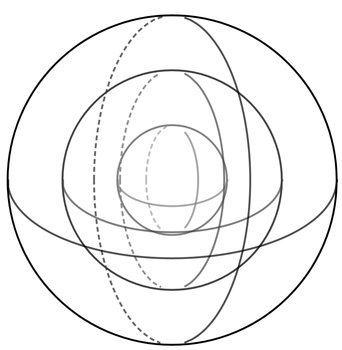
$$A = (\tfrac{\pi e}{8})n$$

$$P \to 2\cos(\tfrac{\pi}{2\sqrt{3}})$$



$$V = (\tfrac{e^2}{6})n$$

$$P \to 2$$



$$V = (\tfrac{e^2}{3})n$$

$$P \to 2\cosh(\tfrac{\pi}{2\sqrt{3}})$$