Versorial Validation: Solution

Setting $p = a + \mathbf{u}$ and $q = b + \mathbf{v}$, we begin by expanding $|pq|^2$:

$$\begin{aligned} |(a+\mathbf{u})(b+\mathbf{v})|^2 &= |(ab - \mathbf{u}\cdot\mathbf{v}) + (a\mathbf{v} + b\mathbf{u} + \mathbf{u}\times\mathbf{v})|^2 \\ &= (ab - \mathbf{u}\cdot\mathbf{v})^2 + \|a\mathbf{v} + b\mathbf{u} + \mathbf{u}\times\mathbf{v}\|^2. \end{aligned}$$

We can FOIL $(ab - \mathbf{u}\cdot\mathbf{v})^2$, and additionally using the relation $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w}$ we can distribute the vector norm above and combine like terms to get

$$\begin{array}{ccccccc} = & a^2b^2 & - & 2ab(\mathbf{u}\cdot\mathbf{v}) & + & (\mathbf{u}\cdot\mathbf{v})^2 \\ + & a^2\|\mathbf{v}\|^2 & + & b^2\|\mathbf{u}\|^2 & + & \|\mathbf{u}\times\mathbf{v}\|^2 \\ + & 2ab(\mathbf{u}\cdot\mathbf{v}) & + & 2a\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) & + & 2b\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}). \end{array}$$

The $\pm 2ab(\mathbf{u}\cdot\mathbf{v})$ terms cancel. The cross product $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, which makes the dot products $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})$ and $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})$ zero, so the magenta terms vanish. For the orange terms, we can use the facts

$$\begin{cases} \mathbf{u}\cdot\mathbf{v} &= \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta, \\ \|\mathbf{u}\times\mathbf{v}\| &= \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta. \end{cases}$$

Using $\cos^2\theta + \sin^2\theta = 1$, the orange terms combine to $\|\mathbf{u}\|^2\|\mathbf{v}\|^2$. So we have

$$a^2b^2 + a^2\|\mathbf{v}\|^2 + b^2\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2$$

$$= (a^2 + \|\mathbf{u}\|^2)(b^2 + \|\mathbf{v}\|^2) = |a+\mathbf{u}|^2|b+\mathbf{v}|^2.$$

In conclusion, we have shown $|pq|^2 = |p|^2|q|^2$.

Hamilton spent about a decade searching for a 3D number system that would model rotations similar to how complex numbers model 2D rotations:

> Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edwin and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."

Orthogonality (the assumption that 1 and i point in orthogonal directions) and multiplicativity are the key properties that let phasors (unit-norm complex numbers of the form $e^{i\theta}$) act as rotations of the complex plane. Hamilton sought a similar system with triples $a + bi + cj$, but to no avail, until one day he realized making i and j anticommute ($ij = -ji$) and ij jut out into a fourth dimension made all the algebra work out:

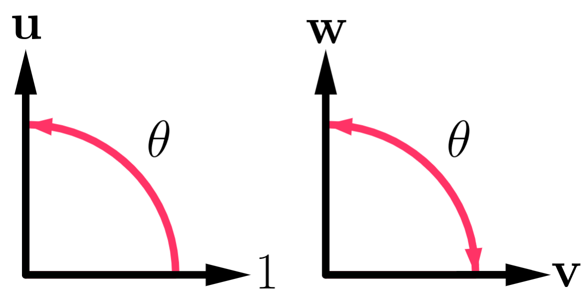
> An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,
>
> $$i^2 = j^2 = k^2 = ijk = -1.$$

(It is a quick exercise to verify this very symmetric equation is equivalent to the usual relations $i^2 = j^2 = k^2 = -1$ and $k = ij = -ji$.) In hindsight, it makes sense three imaginaries are necessary: there are three perpendicular planes of rotation possible in 3D, unlike only one plane of rotation in 2D.

Hamilton may have spent the rest of his life evangelizing quaternions, but they eventually fell out of favor - imagining four dimensions is a hard ask - but Gibbs came along later and cut out the real and imaginary parts of the product of two pure imaginary quaternions and gave us what we now call the dot product and cross product, now standard curriculum today. This story is but a subplot in a larger 'war' waged over various kinds of algebras - other notable names include Gibbs and Heaviside on the side of vectors, and Clifford and Grassman with a multivector generalization of quaternions.

The quaternion is denoted $\mathbb{H}$ in honor of Hamilton. The real and imaginary parts of quaternions are also called the scalar and vector parts. Two quaternions commute ($xy = yx$) if and only if their vector parts are parallel, and they anticommute ($yx = -xy$) if and only if they are perpendicular vectors.

In $\mathbb{H}$, the only square roots of $+1$ form a 'zero-sphere' $S^0 = \{\pm 1\}$, the only square roots of -1 are 3D unit vectors forming the two-sphere S^2, and the versors (unit quaternions) form a hypersphere S^3. All nonzero quaternions have a polar form $r \exp(\theta \mathbf{u})$, where r is the norm, $\mathbf{u}$ is a unit vector, θ is a convex angle $0 \leq \theta \leq \pi$, and Euler's formula applies to $\exp(\theta \mathbf{u})$.



Any unit vector $\mathbf{u}$ can be extended to an orthonormal basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for 3D space (oriented according to the right-hand rule), which extends to a basis $\{1, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for $\mathbb{H}$. If $p = \exp(\theta \mathbf{u})$ then the left-multiplication function $L_p(x) = px$ acts as a rotation by θ in a pair of 2D subspaces, the $1\mathbf{u}$-plane and the $\mathbf{vw}$-plane. The right-multiplication map $R_p(x) = xp$ is the same, but rotates the opposite direction in the $\mathbf{vw}$-plane.

The composition $L_p \circ R_{p^{-1}}$ is conjugation $x \mapsto pxp^{-1}$. These left and right multiplications cancel out in the $1\mathbf{u}$-plane, so when restricted to 3D vectors the effect is rotation around the $\mathbf{u}$-axis by the double angle 2θ. This is how quaternions model 3D rotations. Indeed, they also model 4D rotations: any rotation of four-dimensional space is equivalent to the "bimultiplication" $L_p \circ R_{q^{-1}}$ for some pair of versors p and q.