Twisted Clothesline: Solution

We begin by substituting (k, ℓ) for the indices (so $f_{m+n,2m+3n}$ becomes $f_{k,\ell}$), and then solving for (m, n) by elimination (to rewrite $x^m y^n$):

$$\begin{cases} k = m + n \\ \ell = 2m + 3n \end{cases} \implies \begin{cases} m = 3k - \ell \\ n = \ell - 2k \end{cases}$$

Then the double series becomes

$$\sum_{m,n} f_{m+n,2m+3n} \, x^m y^n = \sum_{k,\ell} f_{k,\ell} \, x^{3k-\ell} y^{\ell-2k}$$

$$= \sum_{k,\ell} f_{k,\ell} \, (x^3 y^{-2})^k (x^{-1} y)^\ell = F(x^3/y^2, y/x).$$

"A generating function is a clothesline on which we hang up a sequence of numbers for display" $-$ *Generatingfunctionology*, page one.