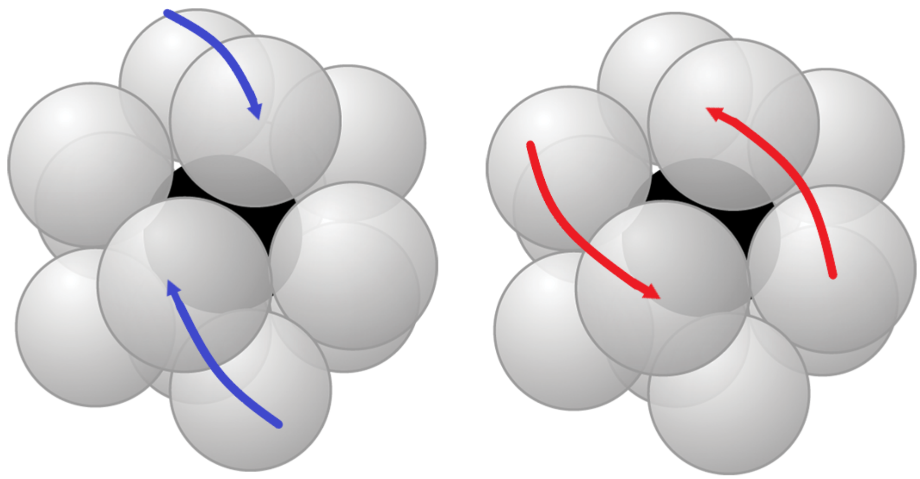
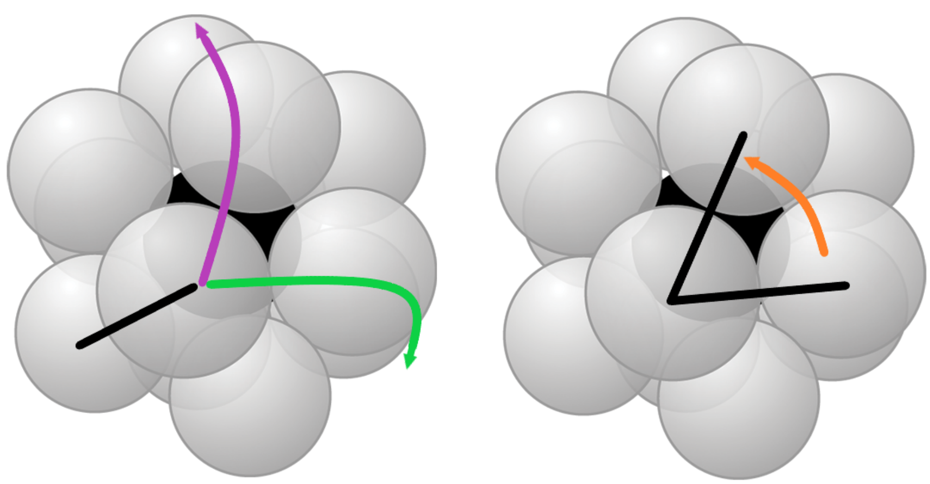
Sporadic Twists: Solution

Any two spheres are either (i) adjacent, (ii) share a common adjacent sphere, or (iii) are antipodal (on opposite sides); in other words, spheres are a distance of 1, 2, or 3 spheres apart from each other. Below we illustrate how if spheres are 2 or 3 spheres apart they can be twisted to become adjacent.



Any adjacent pair of spheres is part of two rings. Every other adjacent sphere is either on one of these two rings or shares a sphere with one of them. Thus, given two adjacent pairs, it is possible to slide the first until it is either the second or at least shares a common sphere with the second. In the latter case, a twist can swivel one pair to become the other.



Let A_1, A_2 and B_1, B_2 denote two pairs of spheres. A sequence of twists turn them into adjacent pairs X_1, X_2 and Y_1, Y_2. Thus, there is a sequence of twists that turns $A_1, A_2 \to X_1, X_2$, one which twists $X_1, X_2 \to Y_1, Y_2$ since they're both adjacent, then we can reverse the sequence turns $B_1, B_2 \to Y_1, Y_2$ to get a sequence of twists that turn $Y_1, Y_2 \to B_1, B_2$.

Putting it all together, we get $A_1, A_2 \to X_1, X_2 \to Y_1, Y_2 \to B_1, B_2$.

The group of permutations generated by twists is the **Mathieu group** M_{12}, the subscript indicating it acts on a set of 12 objects. It is the second smallest among a family of five Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$.

M_{12} is actually **sharply 5-transitive**: given 5 spheres A_1, A_2, A_3, A_4, A_5 and 5 other spheres B_1, B_2, B_3, B_4, B_5 there is a unique permutation of M_{12} which sends $A_i \to B_i$ for $i = 1, 2, 3, 4, 5$. (Our problem merely showed it is 2-transitive!) Besides a full group of *all* permutations on a set (or half that, the group of all even permutations), the Mathieu groups are the *only* permutation groups which act higher than 3-transitively on sets.

In fact, all 2-transitive groups and higher are classified, but the proofs seem to rely on the Classification of Finite Simple Groups, also called the Enormous Theorem. (The Mathieu groups are all simple groups.) The proof of the CFSG is tens of thousands of pages in hundreds of papers by about a hundred mathematicians. It is widely considered the greatest mathematical achievement of the 20th century. A second-generation proof, simplifying the arguments and culling the unnecessary tangents, is in the works.

A **simple** permutation group has no action induced on another (not singleton) set with fewer permutations. For instance, the symmetry group G of a cube can be interpreted as permutations of the 4 space diagonals, but then it has an induced action, involving strictly fewer permutations, on the set of 3 axes, so G is not simple. Contrast with a (regular) icosahedron's symmetry group, which is simple. The Jordan-Hölder theorem describes how all finite groups are built from simple groups.