Slope-Intercept Coordinates: Solution

Substitute $b = y - mx$ into $b = g(m)$ and differentiate with respect to x, get

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{\mathrm{d}m}{\mathrm{d}x}x + m\frac{\mathrm{d}x}{\mathrm{d}x}\right) = g'(m)\frac{\mathrm{d}m}{\mathrm{d}x}$$

by the product and chain rules. Replace $\mathrm{d}y/\mathrm{d}x$ with m and $\mathrm{d}x/\mathrm{d}x$ with 1, the terms m and $-m$ will cancel, then we may divide by $-\mathrm{d}m/\mathrm{d}x$ and replace the m inside $g'(m)$ with $f'(x)$ to get $x = -g'(f'(x))$. Because g' and f' are one-to-one functions, this is sufficient to show $-g'$ and f' are inverse.

In the language of differentials, $y = mx + b$ with the chain rule yields

$$\mathrm{d}y = (\mathrm{d}m)x + m(\mathrm{d}x) + \mathrm{d}b$$

Using $\mathrm{d}y = m\mathrm{d}x$ we may rearrange this to

$$0 \;=\; \mathrm{d}y - m(\mathrm{d}x) \;=\; \mathrm{d}b + x(\mathrm{d}m)$$

or in other words

$$\begin{cases} y = f(x) \\ b = g(m) \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = & m \\ \dfrac{\mathrm{d}b}{\mathrm{d}m} = & -x \end{cases}$$

We may rewrite $\mathrm{d}b/\mathrm{d}m = g'(m)$ as $x = -g'(f'(x))$, just as above.

Functions are **Legendre transformations** of each other when their derivatives are opposite inverses; their graphs are each other's **dual curves**.

The Legendre transformation allows physicists to convert between the so-called Lagrangian and Hamiltonian formulations of classical mechanics, and convert between thermodynamic variables or even create new ones (pressure, volume, temperature, entropy, enthalpy, various energies, and more).