Rational Corollary: Solution

Define the rational function

$$f(A,B) = (I+A)(I-BA)^{-1}(I+B)$$

We want to show f is symmetric, i.e. $f(A,B) = f(B,A)$. First, note

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This formula can be derived by solving a general linear system ($XY = I$) with elimination. It is generalized by Cramer's rule for matrix inverses, which says $X^{-1} = (\det X)^{-1} \operatorname{adj} X$, where $\operatorname{adj} X$ is the adjugate matrix. From here, we can plug two generic 2×2 matrices (thus, a total of eight unknowns) into the rational function f and *laboriously* calculate the result.

Or... we can use a Computer Algebra System like Mathematica:

```
A := {{a11, a12}, {a21, a22}}

B := {{b11, b12}, {b21, b22}}

I2 := IdentityMatrix[2]

(I2 + A) (I2 - B*A)^(-1) (I2 + B)
```

These four lines of input yield the following output:

$$\begin{bmatrix} \dfrac{(1+a_{11})(1+b_{11})}{(1-a_{11}b_{11})} & -1 \\ -1 & \dfrac{(1+a_{22})(1+b_{22})}{(1-a_{22}b_{22})} \end{bmatrix}$$

By inspection we see that swapping as and bs does not change the result.

For scalars there is a geometric sum formula providing the power series expansion $(1-x)^{-1} = 1 + x + x^2 + \cdots$ which converges when $|x| < 1$.

The same power series expansion works for $(I-X)^{-1}$ when X is close enough to the zero matrix. "Closeness" is measured by the **Frobenius norm** (aka **Hilbert-Schmidt norm**) given by $\|X\|^2 = \operatorname{tr}(X^T X) = \sum_{i,j} |x_{ij}|^2$, which is the Euclidean norm with respect to the standard basis for matrices.

If $\|X\| < 1$ then $(I-X)^{-1} = I + X + X^2 + \cdots$. The norm is "submultiplicative," meaning $\|AB\| \leq \|A\|\|B\|$. Thus, when $\|A\|, \|B\| < 1$, we get:

$$f(A,B) = (I+A)(I-BA)^{-1}(I+B)$$

$$= (I+A)\big(I + BA + (BA)^2 + \cdots\big)(I+B)$$

$$\begin{array}{cccccccc} = & I & + & (BA) & + & (BA)^2 & + & \cdots \\ + & A & + & A(BA) & + & A(BA)^2 & + & \cdots \\ + & B & + & (BA)B & + & (BA)^2B & + & \cdots \\ + & AB & + & A(BA)B & + & A(BA)^2B & + & \cdots \end{array}$$

$$\begin{array}{cccccccc} = & I & + & BA & + & BABA & + & \cdots \\ + & A & + & ABA & + & ABABA & + & \cdots \\ + & B & + & BAB & + & BABAB & + & \cdots \\ + & AB & + & ABAB & + & ABABAB & + & \cdots \end{array}$$

This exhibits all possible "words" made from the letters A and B with no consecutive repetitions. Which row a word is located in depends on the first and last letter of the word. Swapping A or B does not change this description, so $f(B,A) = f(A,B)$, for sufficiently "small" matrices at least.

The fundamental theorem of algebra can be used to show that if two polynomials $f(x)$ and $g(x)$ are equal for infinitely many (real or complex) values x, then they are the same polynomial. This generalizes massively: there are sets like the "open ball" $\{X : \|X\| < 1\}$ which are **Zariski dense**, meaning if two rational functions like $f(A,B)$ and $f(B,A)$ are equal on this set of matrices then they are the same rational function.

A theorem due to Krob essentially says any rational identity (like this one) that holds true in any ring (like the ring of matrices) is an algebraic consequence of the geometric sum formula. See *"How would you solve this tantalizing Halmos problem?"* on MathOverflow for more information.

Consider, for instance, the Woodbury formula,

$$(A + UBV)^{-1} = A^{-1} - A^{-1}U(B^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

This doesn't even require A, B to be matrices of the same dimension or U, V to be square matrices! Or consider the simpler Weinstein-Aronszajn identity,

$$\det(I + AB) = \det(I + BA),$$

which does not require A or B to be square. (It is related to a classic abstract algebra exercise: show $1 + ab$ is invertible in a ring iff $1 + ba$ is.)