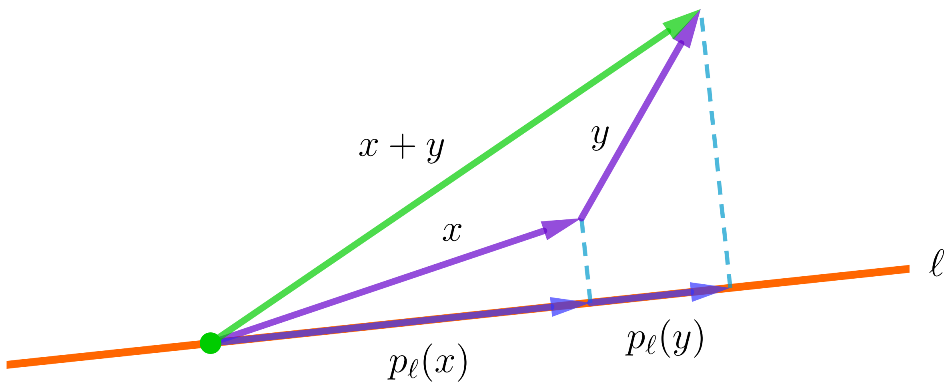
Projector Junction: Solution

Picking an origin on the line ℓ, the projector p_ℓ is a linear map:



Every vector may be decomposed into a sum $v = v_\parallel + v_\perp$ of a parallel and perpendicular component with respect to ℓ, and the projector extracts the parallel component $v_\parallel$ (the perpendicular component $v_\perp$, meanwhile, is called the *rejection* instead of the projection). In other words, if $\ell^\perp$ is the line perpendicular to ℓ through the chosen origin, then p_ℓ is characterized by

$$p_\ell(v) = \begin{cases} v & \text{if } v \text{ is on } \ell \\ 0 & \text{if } v \text{ is on } \ell^\perp \end{cases}$$

Since p_ℓ is linear, it may be represented as a matrix. In fact, if we pick either of the two unit vectors $\pm u$ in ℓ (represented as column vectors), then the projector is $p_\ell = uu^T$. To see this, note the u^T in uu^T ensures the kernel (aka nullspace) is $\ell^\perp$ and the u in uu^T ensures the image (aka columnspace or range) is ℓ. More specifically, uu^T has the same characterization as p_ℓ since $(uu^T)v = (u \cdot v)u$ and $u \cdot v$ is 0 if v is on $\ell^\perp$ or is λ if $v = \lambda u$ is on ℓ.

For the diagram with lines k and ℓ, it will be convenient to choose their intersection as the origin and use m and n for our x and y axes. Then the lines are at an angle $\phi = \theta/2$ to m, so we can choose unit vectors $\left[\begin{smallmatrix} \cos\phi \\ \pm\sin\phi \end{smallmatrix}\right]$ on them. The corresponding projectors are then

$$p_k = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}, \text{ and}$$

$$p_\ell = \begin{bmatrix} \cos\phi \\ -\sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{bmatrix}.$$

The products $p_k p_\ell$ and $p_\ell p_k$ may be calculated as

$$p_k p_\ell = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{bmatrix}$$

$$= (\cos^2\phi - \sin^2\phi) \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ \sin\phi\cos\phi & -\sin^2\phi \end{bmatrix},$$

$$p_\ell p_k = \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}$$

$$= (\cos^2\phi - \sin^2\phi) \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ -\sin\phi\cos\phi & -\sin^2\phi \end{bmatrix}.$$

So, by $\theta = 2\phi$ and the double angle formulas for cos, the symmetrization is

$$\frac{1}{2}(p_k p_\ell + p_\ell p_k) \;=\; (\cos^2\phi - \sin^2\phi) \begin{bmatrix} \cos^2\phi & 0 \\ 0 & -\sin^2\phi \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} \frac{1}{2}(\cos\theta + 1) & 0 \\ 0 & \frac{1}{2}(\cos\theta - 1) \end{bmatrix}$$

Since $p_m = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $p_n = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, this finally means

$$a(\theta) = \tfrac{1}{2}\cos\theta(\cos\theta + 1), \qquad b(\theta) = \tfrac{1}{2}\cos\theta(\cos\theta - 1).$$

This calculation is relevant in a certain kind of algebraic structure: Jordan algebras, which were a first attempt to formalize quantum observables.

A **Jordan algebra** is a power-associative algebra (where the associative property may fail, but powers like $x^3 = (xx)x = x(xx)$ are still well-defined) where left-multiplication and right-multiplication by powers commute, i.e. $x^m(yx^n) = (x^m y)x^n$ for all elements x, y in the algebra. (The Jordan identity $(xy)x^2 = x(yx^2)$ is a special case, and implies all the other cases.)

The **formally real** Jordan algebras are those where no sum of nonzero squares is zero (just like the real numbers). As with many other algebraic structures, we can define homomorphisms, ideals, direct sums and the like.

Simple Jordan algebras are those with no proper nonzero ideals, or equivalently no proper nonzero homomorphic images. Every formally real Jordan algebra is a direct sum of simple ones. All simple formally real Jordan algebras are of Clifford type or matrix type. The latter are $n \times n$ self-adjoint matrices over real numbers, complex numbers, or quaternions (or octonions for $n \leq 3$), but instead of using the usual matrix multiplication they use the (normalized) **anticommutator** $\{A, B\} := \frac{1}{2}(AB + BA)$.

The **spectral theorem** implies any real symmetric (i.e. self-adjoint) matrix is a linear combination of orthogonal projectors (which are not only orthogonal geometrically, but algebraically as well: $p_k p_\ell = 0$ if k and ℓ are perpendicular lines). This problem shows how to find this decomposition for the anticommutator of two projectors.