Polarization: Solution

First we extend the vanishing condition. Substitute $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ into the vanishing condition, and then distribute (aka "FOIL") with multlinearity for

$$\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_2, \mathbf{b}) + \phi(\mathbf{a}_2, \mathbf{b}, \mathbf{a}_1, \mathbf{b}) = 0$$

The terms $\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_1, \mathbf{b})$ and $\phi(\mathbf{a}_2, \mathbf{b}, \mathbf{a}_2, \mathbf{b})$ are zero so do not appear. By the symmetry condition, the two remaining terms are equal, so $\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_2, \mathbf{b}) = 0$.

Similarly, substituting $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ into $\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_2, \mathbf{b}) = 0$ gives

$$\phi(\mathbf{a}_1, \mathbf{b}_2, \mathbf{a}_2, \mathbf{b}_1) = -\phi(\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2)$$

Thus, swapping the second and fourth arguments changes the sign. If we had instead substituted $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ first and $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ second we would have found swapping the first and third arguments also changes the sign.

In conclusion, ϕ is fully **antisymmetric**: swapping any two of its arguments changes its sign. This also forces ϕ to be **alternating**: if any two of its arguments are equal, ϕ vanishes (equals 0). This is because if two arguments are equal, then swapping them changes the sign but also does nothing, and the only scalar value ϕ satisfying $\phi = -\phi$ is $\phi = 0$.

Finally, for $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, we can use basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to write

$$\begin{cases} \mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \\ \mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \\ \mathbf{d} = d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{e}_3 \end{cases}$$

which means $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{\ell=1}^{3} a_i b_j c_k d_\ell \; \phi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_\ell)$.

Two of i, j, k, ℓ must be equal by the pigeonhole principle, which means all of the summands above are 0, forcing $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = 0$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

This solution shows how the two-vector **Lagrange identity**

$$\|\mathbf{a} \times \mathbf{b}\|^2 = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

in three dimensions implies the four-vector **Binet-Cauchy identity**

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$= \det \begin{pmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{pmatrix} = \det \begin{pmatrix} | & | \\ \mathbf{a} & \mathbf{b} \\ | & | \end{pmatrix}^T \begin{pmatrix} | & | \\ \mathbf{c} & \mathbf{d} \\ | & | \end{pmatrix}$$

of which the Lagrange identity is a special case: set $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ to be the difference between the left and right sides of Binet-Cauchy, then show $\phi \equiv 0$.

The situation is different in higher dimensions - in four dimensions, for instance, there is a nonzero alternating form satisfying all four properties:

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \det \begin{pmatrix} | & | & | & | \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ | & | & | & | \end{pmatrix}.$$

Even more generally, in n dimensions the set of all multilinear alternating forms of k variables forms an $\binom{n}{k}$-dimensional vector space called the **exterior power** $\Lambda^k \mathbb{R}^n$ (or technically its dual, depending on definitions).

Besides the pigeonhole principle, this solution uses **polarization**, a technique for converting between homogeneous multivariable polynomials of degree d and multilinear forms of d variables. The simplest nontrivial case is converting between quadratic and bilinear forms, as seen in any of the many **polarization identities** relating squared norms and inner products:

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\|^2.$$

The relation $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$ tells us how to write norms in terms of dot products and leads to this identity by substituting $\mathbf{v} = \mathbf{a} + \mathbf{b}$, and conversely this identity tells us how to rewrite dot products in terms of norms.

Another equivalent polarization identity does the same trick,

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{4} \left(\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 \right),$$

and is the antisymmetrized sibling of the **parallelogram law**

$$2 \left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \right) = \|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2.$$

Exercise 3.7 of *The Cauchy-Schwarz Master Class* challenges the reader to upgrade the n-dimensional version of the two-vector Lagrange identity to the n-dimensional version of the four-vector Binet-Cauchy identity,

$$(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) = \sum_{k<\ell} \begin{vmatrix} a_k & b_k \\ a_\ell & b_\ell \end{vmatrix} \begin{vmatrix} c_k & d_k \\ c_\ell & d_\ell \end{vmatrix}.$$

(The text is a dedicated compendium of applications and offshoots of the **Cauchy-Schwarz inequality** $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$, which itself follows from polarizing the positivity condition $\|\mathbf{a} - \mathbf{b}\|^2 \geq 0$.)

Surprisingly, the text's hint to use polarization seems erroneous, since the difference between the left and right sides of Binet-Cauchy satisfy the four properties given in the problem (which are the algebraic features of the form that allow for polarization) but we saw for $n \geq 4$ there are nonzero forms.