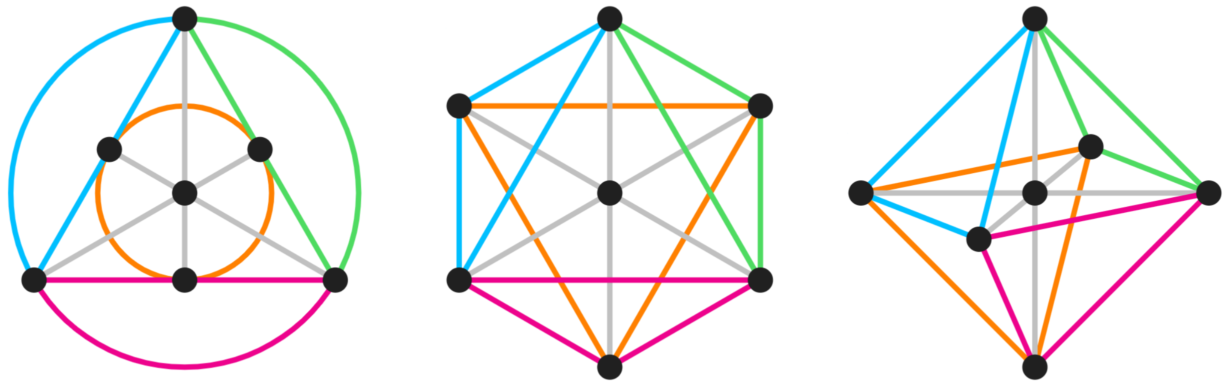
Perspective Shift: Solution



First, stretch out the triangle edges' midpoints, and tighten the circular arcs until they're taut, to get a hexagon. Then, pull those same three vertices out of the page to get a "triangular antiprism," i.e. an octahedron!

Around all three axes there is fourfold rotational symmetry. In fact, we can color opposite faces the same color using four colors, and then there is exactly one rotational symmetry for each of the $4! = 24$ permutations of these colors!

The graph of this problem, minus its outer circle, is a famous depiction of the **Fano plane** $\mathbb{F}_2\mathbb{P}^2$ studied in finite geometry.

Finite geometry studies finite sets with combinatorial structures satisfying axioms from geometry, often modeled with equations involving finite fields (number systems with only finitely many numbers). In this context, the axioms come from projective geometry, a branch of geometry historically influenced by the development of perspective drawing in Renaissance art.

The real projective plane $\mathbb{RP}^2$ is the 2D space whose points represent all possible 1D subspaces of 3D Euclidean space $\mathbb{R}^3$. Every 1D subspace of $\mathbb{R}^3$ can be represented by a pair of unit vectors, so $\mathbb{RP}^2$ is like the sphere S^2 but where antipodal points $\pm\mathbf{v}$ of S^2 count as the same point of $\mathbb{RP}^2$. The real projective plane, like the Klein bottle, is impossible to embed in $\mathbb{R}^3$ without self-intersection, although immersions are possible (notably Boy's surface).

The Fano plane $\mathbb{F}_2\mathbb{P}^2$ is defined the same way, but uses the finite field $\mathbb{F}_2$ instead of the real numbers $\mathbb{R}$. The term field means there is addition and multiplication satisfying commutativity, distributivity and associativity, and $\mathbb{F}_2$ in particular has only two numbers, an additive identity called 0 and a multiplicative identity called 1. Arithmetic is what you'd expect, except $1+1 = 0$.

This means $\mathbb{F}_2^3$ has $2^3 = 8$ vectors (a, b, c), and every 1D subspace contains exactly one nonzero vector so there are seven elements of the Fano plane $\mathbb{F}_2\mathbb{P}^2$, corresponding to the seven vertices of the graphical depiction. Every 2D subspace of $\mathbb{F}_2^3$ (called a line of $\mathbb{F}_2\mathbb{P}^2$) contains exactly three 1D subspaces, which suggests drawing every possible edge between seven vertices (i.e., a complete graph K_7) and coloring the 3-cycles which are projective lines. The usual triangular Fano plane depiction is missing edges, unfortunately.
The fact that the Fano plane's symmetry group $\mathrm{PGL}_3\mathbb{F}_2$ has

$$(2^3 - 1)(2^3 - 2)(2^3 - 2^2) \;=\; 168 \;=\; 24 \cdot 7$$

elements means there ought to be a picture which captures its sevenfold symmetry (which is not apparent in the octahedral picture). This picture can be obtained by considering the exceptional isomorphism $\mathrm{PGL}_3\mathbb{F}_2 \cong \mathrm{PSL}_2\mathbb{F}_7$ and the metacyclic subgroup of $\mathrm{PSL}_2\mathbb{F}_7$ (whose Möbius transformations acting on the projective line $\mathbb{F}_7\mathbb{P}^1$ are affine transformations fixing $1/0 = \infty$). It can also be a helpful mnemonic for octonion multiplication! Octonions are an eight-dimensional nonassociative number system generalizing quaternions.