Pentagonal Peculiarity: Solution

Let r be the number of dots in the bottom row and d on the right diagonal.

▶ If $r > d$, we can pour the right diagonal into a new row. This increases the number of rows by one. The new row will have strictly less dots than the one above, so all rows have distinct numbers of dots, *unless* a dot from the *original* last row gets poured into the *new* last row! This will happen if the last row and right diagonal share a corner dot, in which case the new row will fail to have fewer dots than the row above precisely if $r = d + 1$.

In this case, the number of dots is

$$\begin{aligned} n &= r + (r+1) + \cdots + (r + r - 2) \\ &= (r-1)r + \frac{(r-2)(r-1)}{2} \\ &= \frac{(3r-2)(r-1)}{2} = \frac{(3d+1)d}{2}. \end{aligned}$$

▶ If $r \leq d$, we can scoop the last row into the right diagonal. The rows will still have distinct numbers of dots. This decreases the number of rows by one, *unless* we scoop a dot back into the last row! This will happen if $r = d$ and again a corner dot is shared by the last row and right diagonal.

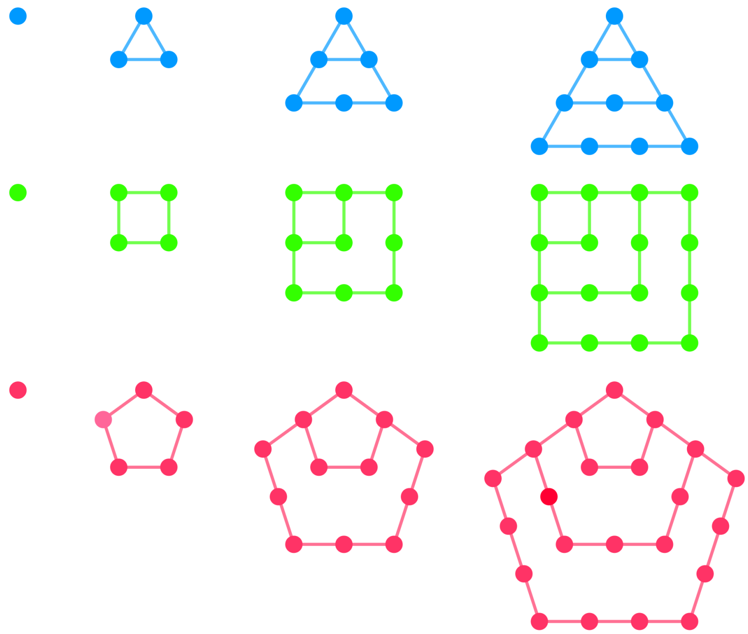
In this case the number of dots is

$$\begin{aligned} n &= r + (r+1) + \cdots + (r + r - 1) \\ &= r^2 + \frac{r(r-1)}{2} = \frac{(3r-1)r}{2}. \end{aligned}$$

Defining the kth **pentagonal number** $g(k) = (3k-1)k/2$, the second case has $n = g(r)$ and the first case has $n = g(-d)$. Note g is a one-to-one function since $g(0) = 0$ and $g(k) < g(-k) < g(k+1)$ for all $k > 0$. Thus, either one exception or the other can occur, depending on n, but not both.

In conclusion, the pouring-and-scooping procedure pairs the even-row diagrams with the odd-row diagrams, with at most one exception, so $|E - O| = 1$.

The triangular numbers, square numbers, and pentagonal numbers are so-named because they count dots in series of expanding geometric figures:



This has applications to expanding a certain infinite product into a series,

$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{n=0}^{\infty} \square\, q^n$$

When we expand out the product, infinitely, the resulting terms are of the form $(-1)^r q^{m_1+\cdots+m_r}$ for distinct exponents $m_1, \cdots, m_r$. These terms correspond to diagrams with n dots - specifically, with m_1 in the first row, m_2 in the second row, and so on. Each diagram contributes ± 1 to the coefficient $\square$ depending on whether the number of rows is even or odd.

Thus, $\square = E - O$. We've seen this is 0 except when $n = g(k)$ is a generalized pentagonal number. In both kinds of exceptions we examined, the number of rows was k (either $k = d$ when $r = d+1$ or $k = r$ when $r = d$). Therefore

$$\prod_{m=1}^{\infty}(1-q^m) \;=\; \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

This is the **Pentagonal Number Theorem**.