Pair of Pairs: Solution

The number $\binom{n}{2}$ counts how many unordered pairs $\{a,b\}$ there are with a, b distinct numbers drawn from $\{1, \cdots, n\}$. Furthermore, the expression

$$\binom{\binom{n}{2}}{2}$$

counts how many pairs of pairs $\{\{a,b\},\{c,d\}\}$ there are with a, b, c, d from $\{1, \cdots, n\}$, with $a \neq b$ and $c \neq d$ as well as $\{a,b\} \neq \{c,d\}$. There are two cases: the sets $\{a,b\}$ and $\{c,d\}$ either share one number in common, or none.

In the first case, there are $\binom{n}{3}$ ways to pick three distinct numbers $\{a,b,c\}$ from $\{1, \cdots, n\}$, and then 3 ways to construct a pair of pairs out of them:

$$\{\{a,b\},\{a,c\}\}, \qquad \{\{a,b\},\{b,c\}\}, \qquad \{\{a,c\},\{b,c\}\},$$

each corresponding to a choice of one of a, b, c (assume $a < b < c$ so these subsets are distinguishable) to be repeated. In the second case, there are $\binom{n}{4}$ ways to pick four distinct numbers $\{a,b,c,d\}$ (again assume $a < b < c < d$), and then there are four ways to partition these four into two pairs of pairs:

$$\{\{a,b\},\{c,d\}\}, \qquad \{\{a,c\},\{b,d\}\}, \qquad \{\{a,d\},\{b,c\}\}.$$

Putting it all together, we can write this as an equation

$$\binom{\binom{n}{2}}{2} = 3\binom{n}{3} + 3\binom{n}{4}.$$

Alternatively, we could have used the formula $\binom{n}{k} = \frac{n}{k}\frac{n-1}{k-1}\frac{n-2}{k-2}\cdots$ (with k fractions being multiplied) to verify the identity algebraically:

$$\frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} - 1\right)}{2} = 3\,\frac{n(n-1)(n-2)}{6} + 3\,\frac{n(n-1)(n-2)(n-3)}{24}.$$