Noncommutative Calculus: Solution

We can work "mod t^4," meaning ignore any powers of t higher than t^3.

The left-hand side $\exp(tX)\exp(tY)$ is

$$\Big(1 + tX + \tfrac{1}{2}t^2X^2 + \tfrac{1}{6}t^3X^3 + \cdots\Big)\Big(1 + tY + \tfrac{1}{2}t^2Y^2 + \tfrac{1}{6}t^3Y^3 + \cdots\Big) =$$

$$1 + t(X+Y) + \tfrac{1}{2}t^2\big(X^2 + 2XY + Y^2\big) + \tfrac{1}{6}t^3\big(X^3 + 3X^2Y + 3XY^2 + Y^3\big) + \cdots$$

On the other hand, the right-hand side $\exp(tZ_1 + t^2Z_2 + t^3Z_3 + \cdots)$ is

$$1 + (tZ_1 + t^2Z_2 + t^3Z_3 + \cdots) + \tfrac{1}{2}(tZ_1 + t^2Z_2 + \cdots)^2 + \tfrac{1}{6}(tZ_1 + \cdots)^3 + \cdots$$

$$= 1 + tZ_1 + \tfrac{1}{2}t^2(Z_1^2 + 2Z_2) + \tfrac{1}{6}t^3(Z_1^3 + 3Z_1Z_2 + 3Z_2Z_1 + 6Z_3) + \cdots$$

Equating coefficients of t gives $Z_1 = X + Y$, and equating coefficients of $\frac{1}{2}t^2$ gives $Z_1^2 + 2Z_2 = X^2 + 2XY + Y^2$: substituting Z_1 into the latter we can solve $Z_2 = \frac{1}{2}(XY - YX)$. Equating coefficients of $\frac{1}{6}t^3$ and substituting gives

$$(X+Y)^3 + \tfrac{3}{2}(X+Y)(XY - YX) + \tfrac{3}{2}(XY - YX)(X+Y) + 6Z_3$$

$$= X^3 + 3X^2Y + 3XY^2 + Y^3.$$

Distributing, subtracting, cancelling, and dividing by 6 gives

$$Z_3 \;=\; \tfrac{1}{12}\big(X^2Y - 2XYX + YX^2 \;+\; Y^2X - 2YXY + XY^2\big).$$

The noncommutative polynomials $Z_k(X,Y)$ may be expressed much more compactly using the **commutator** operation $[X,Y] := XY - YX$:

$$\begin{aligned} Z_1 &= X + Y, \\ Z_2 &= \tfrac{1}{2}[X,Y], \\ Z_3 &= \tfrac{1}{12}[X,[X,Y]] + \tfrac{1}{12}[Y,[Y,X]], \\ Z_4 &= \tfrac{1}{24}[Y,[X,[Y,X]]], \\ &\cdots \end{aligned}$$

A formula of Dynkin says that in $Z_n(X,Y)$ the coefficient of

$$[\underbrace{X,[X,\cdots[X}_{r_1},[\underbrace{Y,[Y,\cdots[Y}_{s_1},\cdots[\underbrace{X,[X,\cdots[X}_{r_n},[\underbrace{Y,[Y,\cdots Y}_{s_n}]\cdots\cdots]$$

is $\frac{(-1)^{n-1}}{n}$ times the reciprocal of $(r_1+\cdots+r_n+s_1+\cdots+s_n)r_1!\cdots r_n!s_1!\cdots s_n!$.

The full solution $Z(X,Y)=\sum_{n=0}^{\infty} Z_n(X,Y)$ to $\exp(X)\exp(Y)=\exp Z$ (so, when $t=1$) is known as the **Baker-Campbell-Hausdorff formula**.

Beginning with Klein's Erlangen Program at the turn of the 20th century, mathematicians began studying the geometry of homogeneous spaces from the perspective of symmetry groups. ("Homogeneous," here, means no point in space is more special than any other point in space.) The symmetry of a sphere, for example, is the matrix group SO(3) of 3D rotation matrices.

Born from this was an interest in the action of continuous symmetry groups, called **Lie groups**, particularly when it came to solving differential equations describing motion and dynamics. Lie's idea was to think about so-called *infinitessimal* symmetries, or in other words the derivatives of animations of symmetry (imagine, for instance, an animation of a sphere rotating around an axis). The infinitessimal symmetries form a **Lie algebra**.

The BCH formula implies it is possible to reconstruct the composition operation of the Lie group from the bracket operation of the Lie algebra. This is one of many results of a larger Lie-group−Lie-algebra correspondence. More generally, the composition, conjugation, and commutator operations of a Lie group correspond respectively (via differentiation) to the addition, adjoint, and bracket operations in the corresponding Lie algebra.