# Non-Commutative Calculus

The exponential function is often given by the infinite power series formula

$$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!} = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \cdots$$

It satisfies $\exp(X+Y) = \exp(X)\exp(Y)$ when $X$ and $Y$ commute (meaning $XY = YX$) but generally not otherwise, e.g. for matrices it is usually false.

But even for noncommuting $X$ and $Y$ there is an infinite series for which

$$\exp(tX)\exp(tY) = \exp\big(tZ_1 + t^2Z_2 + t^3Z_3 + \cdots\big)$$

where each $Z_k$ is a degree $k$ *noncommutative* polynomial of $X$ and $Y$ (and we assume $t$ commutes with everything). In particular, for matrices the $Z$-series can be evaluated for scalar values $t$ small enough relative to $X$ and $Y$.

For example, $Z_1 = X + Y$ (unsurprisingly) and $Z_2 = \frac{1}{2}(XY - YX)$.

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**Problem**. Find the noncommutative polynomial $Z_3(X, Y)$.

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Submit your solution online by scanning QR code and filling out the form, or submit at

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A photo of handwritten work is fine. You can also turn in solutions physically at the UNO math department's mail room (located on the second floor of the Durham Science Center).