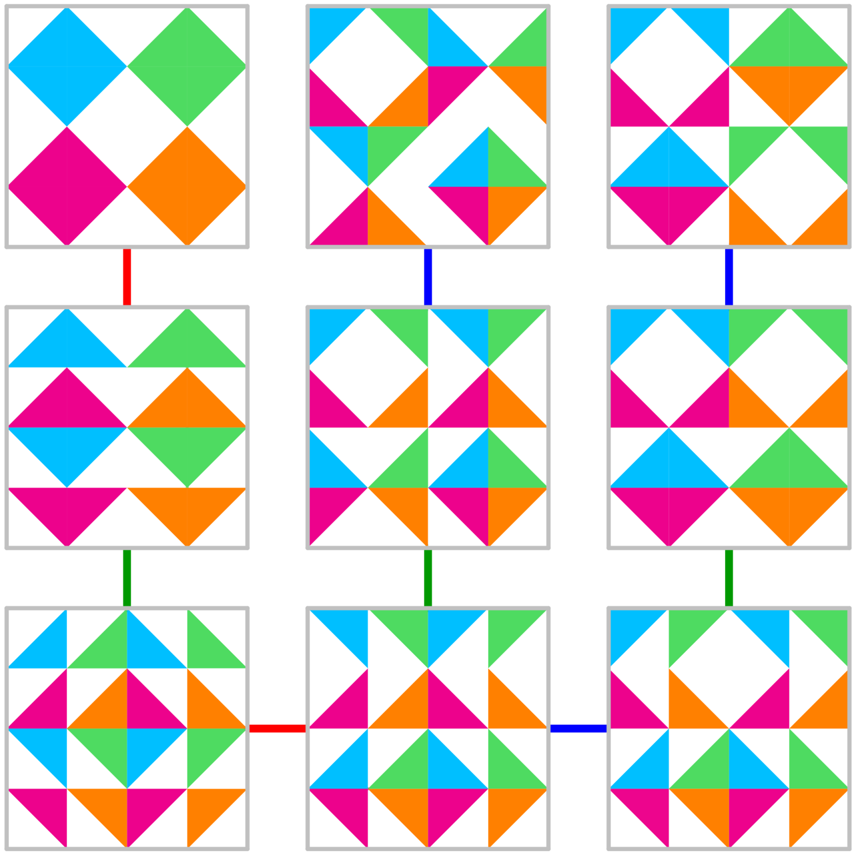
Kaleidoscopic Diamonds: Solution

There are tons of ways to go between the top three diagrams (indeed, infinitely many if we allow backtracking and going in circles between diagrams).



Above is a fairly compact web showing how they can all be connected, where:

- red lines mean row swap,
- green lines mean column swap,
- blue lines mean block swap.

The tiles are also color-coded so that all of the sixteen tiles have a unique combination of color and orientation. Can you see which swaps take place?

The full problem, created and hosted online by Steven H. Cullinane, a self-described finite geometry enthusiast, includes many more diagrams. There is also a harder version challenging the player to turn a pair of diagrams into another pair if the moves apply to both diagrams simultaneously!

Finite geometry studies what happens when the axioms and operations of geometry apply to finite sets of points - indeed, the coordinates and equations of finite geometry replace the real number system with finite fields (number systems which have the usual four arithmetic operations $+, -, \times, \div$ but only finitely many numbers, like the integers mod p where p is prime).

Let G be the group of permutations of the sixteen positions generated from swapping rows, columns, or blocks. Cullinane published a "Diamond theorem" which says any permutation of G applied to the first diamond figure always results in another figure with symmetry involving ordinary rotation, reflection and/or color-swapping the white and gray of tiles. The number of diagrams attainable from the first one is $24 \cdot 35 = 840$.

G has a a subgroup H (with 1/16th of the permutations of G) which has two other interesting incarnations: (a) the group A_8 of all even permutations of $\{1, \cdots, 8\}$, those that arise from an even number of swaps, and (b) the group $\mathrm{GL}_4\mathbb{F}_2$ of all invertible 4×4 matrices with bit entries (0 or 1) and bitwise arithmetic (where $+$ is logical `XOR` and $\times$ is logical `AND`). The full group $G \cong \mathrm{Aff}_4\mathbb{F}_2$ is equivalent to all affine transformations of $\mathbb{F}_2^4$ (four-dimensional space with bit coordinates), and has $8! \cdot 8 = 322\,560$ permutations.