Interesting Asymptotic: Solution

By repeatedly differentiating $\ln(1+x)$ we can reasonably guess, and then prove, a formula for its nth derivative, and then determine the coefficients of its Taylor-Maclaurin power series. Alternatively, we can find the definite integral of the geometric series for $1/(1+t)$ from 0 to x.

Either way, we arrive at the so-called Newton-Mercator series:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Thus we may rewrite $\frac{1}{e}(1+\frac{1}{n})^n$ as powers of e and then

$$\begin{aligned}
\frac{1}{e}\left(1+\frac{1}{n}\right)^n &= \exp\left[-1 + n\ln\left(1+\frac{1}{n}\right)\right] \\
&= \exp\left[-1 + n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right] \\
&= \exp\left(-\frac{1}{2n} + \frac{1}{3n^2} - \cdots\right) \\
&= 1 + \left(-\frac{1}{2n} + \frac{1}{3n^2} - \cdots\right) + \frac{1}{2!}\left(-\frac{1}{2n} + \cdots\right)^2 + \cdots \\
&= 1 - \frac{1}{2n} + \frac{1}{3n^2} + \frac{1}{8n^2} + \cdots
\end{aligned}$$

And therefore we conclude $a = -\frac{1}{2}$, $b = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$, or in other words

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx 1 - \frac{1}{2n} + \frac{11}{24n^2}.$$