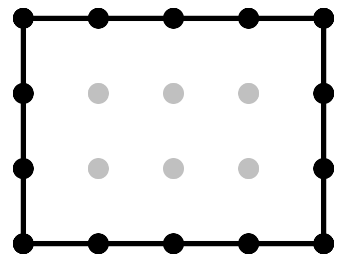
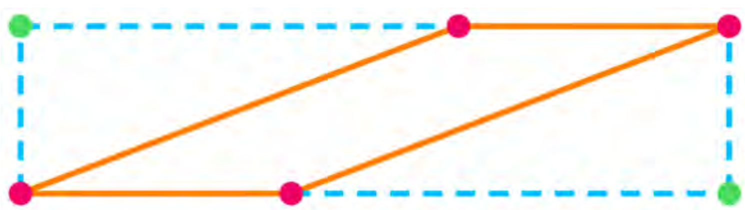
# Fenced In: Solution

For an upright rectangle with length $L$ and width $W$, each lengthwise edge has $L+1$ grid points, each widthwise edge has $W+1$ grid points, and the interior has $(L-1)(W-1)$ grid points. See below for a small example:



We can extend the orange-magenta parallelogram to a larger rectangle by adding on blue-green right triangles to its upper left and lower right corners:



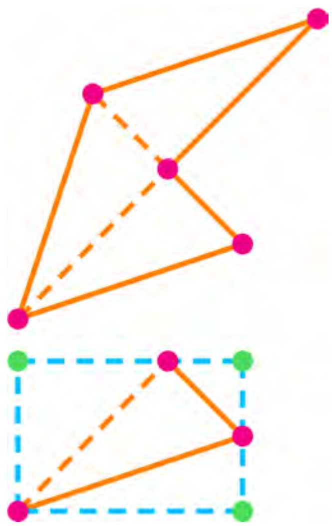
The larger rectangle has dimensions $89 \times 377$. If the two blue-green triangles were combined into a smaller triangle, it would have dimensions $89 \times 233$.

Are there any points on the orange diagonals? The slope $m = 89/233$ of the left diagonal is already in lowest terms, so cannot be expressed as a fraction $b/a$ with smaller numbers, and so there is no point $(a, b)$ on it between its endpoints. Any point on the right diagonal would correspond to one on the left diagonal, so there are no points on the right diagonal either.

The number of grid points within the parallelogram is therefore the difference between the numbers within the larger and smaller rectangles:

$$(88)(376) - (88)(232) \;=\; 88 \cdot 144 \;=\; \mathbf{12672}.$$

What about counting interior grid points of an arbitrary polygon?



**Pick's Theorem** says the area $A$ of a polygon with integer-coordinate vertices can be expressed in terms of the number $i$ of interior grid points and the number $b$ of grid points on the boundary:

$$A = i + \tfrac{1}{2}b - 1. \qquad \text{(PT)}$$

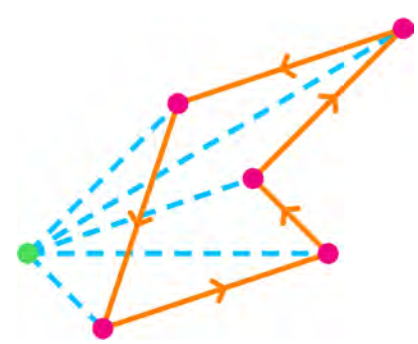
We can divide any polygon into triangles and add their PT equations up to get PT for the polygon. PT is true for rectangles because $A = LW$, $i = (L-1)(W-1)$, and $b = 2L+2W$, and right triangles (by halving rectangles), so it's true for triangles in general (by fitting them in bounding boxes alongside right triangles).

Meanwhile, the **Shoelace Formula** says the area of a polygon with vertices $(x_1, y_2)$, $(x_2, y_2)$, $\cdots$, $(x_n, y_n)$ (write $(x_{n+1}, y_{n+1}) = (x_1, y_1)$, too) is given by

$$A \;=\; \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{vmatrix} \;\stackrel{\text{def}}{=}\; \sum_{k=1}^{n} \frac{1}{2} \underbrace{\begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix}}_{x_k y_{k+1} - x_{k+1} y_k} .$$

The formula is so-named because in the $2 \times n$ array above if we draw lines to pair up the $x$s and $y$s that get multiplied, it looks like we're lacing them up.

Each vertex may be interpreted as a vector, and then any edge of the polygon, alongside the pair of vectors to its endpoints, forms a triangle.



If we orient the edges around the polygon in a loop, then we can say the triangles have positive or negative area as a appropriate, and then the sum of their signed areas is the area of the polygon!

(Note if a triangle has two edges meeting at the origin, interpreted as column vectors $\mathbf{a}$ and $\mathbf{b}$, its signed area is half of the determinant $\det(\mathbf{a}\ \mathbf{b}) = \left|\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right|$.)

Using $x_k y_{k+1} - x_{k+1} y_k = x_k(y_{k+1} - y_k) - (x_{k+1} - x_k)y_k$ we may rewrite the summands of the shoelace formula. In the limit the formula becomes

$$A = \frac{1}{2}\sum \left(x\Delta y - y\Delta x\right) \quad \longrightarrow \quad \frac{1}{2}\oint x\,\mathrm{d}y - y\,\mathrm{d}x.$$

In vector calculus this contour integral is a special case of **Green's theorem**, which is itself a special case of the **curl theorem**. This is the theoretical basis for the real-life **planimeter** tool used to calculate areas.