Cusp of Crying: Solution

The cusp $(0,1)$ has two tangent lines. By symmetry, their slopes are $\pm m$ for some m. Their slopes will help us find the angle. Assume the curve is parametrized by $(x(t), y(t))$ leading up to the cusp.

Squaring $r = e^{y-1}$ yields $x^2 + y^2 = e^{2y-2}$. Differentiating and halving yields

$$xx' + yy' = y' e^{2y-2}$$

Collect like terms for y' on the right, replace e^{2y-2} with $x^2 + y^2$, then divide:

$$1 = \left(\frac{x^2 + y^2 - y}{x} \right) \frac{y'}{x'} = \left(x + y\,\frac{y-1}{x} \right) \frac{y'}{x'}.$$

The ratio $(y-1)/x$ is the slope of the secant line from $(0,1)$ to (x,y), and y'/x' is the slope of the tangent line at (x,y). Therefore, in the limit $(x,y) \to (0,1)$,

$$1 = \big(0 + 1 \cdot m\big)\, m = m^2.$$

Thus, $m = \pm 1$, and the cusp is a right angle ($\angle = 90°$).

The exponential function has the globally convergent power series

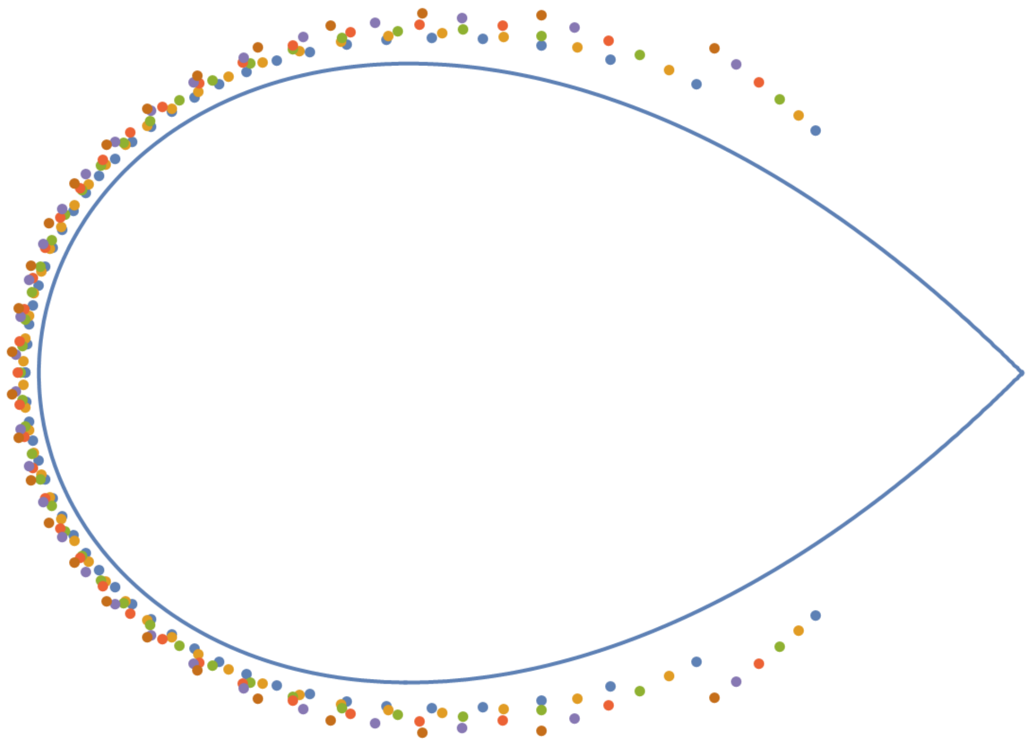
$$\exp z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

The truncations of this series to the first so many terms,

$$\exp_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!},$$

are polynomials and therefore have complex roots. But $\exp z$ itself has no complex roots! Thus it's no surprise the roots of $\exp_n(z)$, as $n \to \infty$, expand outward without bound. And yet, they still approach a certain shape.

The roots of $\exp_n$ divided by n tend towards this curve:



This is the **Szëgo curve** defined by $|ze^{1-z}| = 1$ in the complex plane. By rearranging this to $|z| = |e^{z-1}|$ and setting $z = x + yi$ we may rewrite it in Cartesian coordiantes as $\sqrt{x^2 + y^2} = e^{x-1}$; flipping this across the diagonal line $y = x$ leaves us with the teardrop curve of this problem.