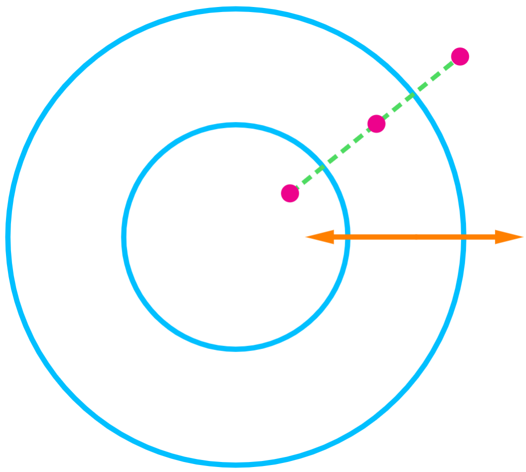
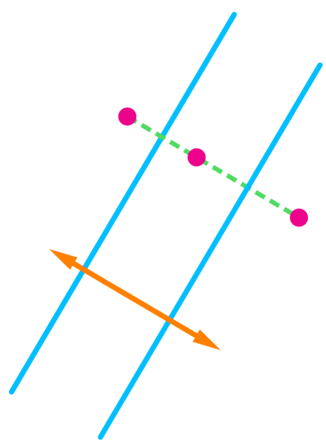
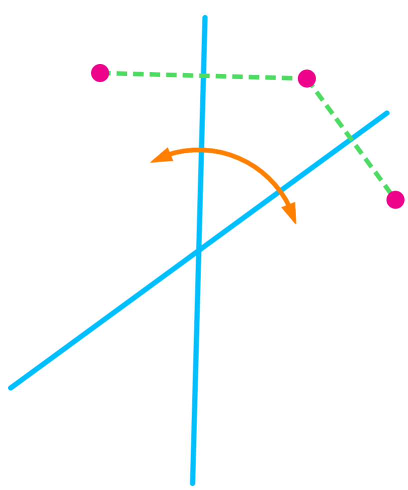
Celestial Shifting: Solution



Concentric Circles



Parallel Lines



Intersecting Lines

Following the hint, we summarize our investigation thusly:

Concentric circles. If the distance of a point from a radius r circle's center is denoted x, the effect of circle inversion is $x \mapsto r^2/x$. Thus, if we compose a circle inversion (radius r) followed by another circle inversion (radius R, concentric), the combined effect is $x \mapsto R^2/(r^2/x)$, or simply $x \mapsto (R/r)^2 x$.

In effect, we can stretch space from any point by any multiplicative factor (or shrink it towards any point, if we use a factor smaller than 1) if we compose inversions across concentric circles with the appropriate radii.

Parallel lines. Composing inversions across two parallel lines has the effect of sliding all points in a direction perpendicular to the lines by twice the distance between the lines.

Intersecting lines. A similar geometric argument shows composing inversions across two intersecting lines has the effect of rotating all points around the point of intersection by twice the angle between the lines.

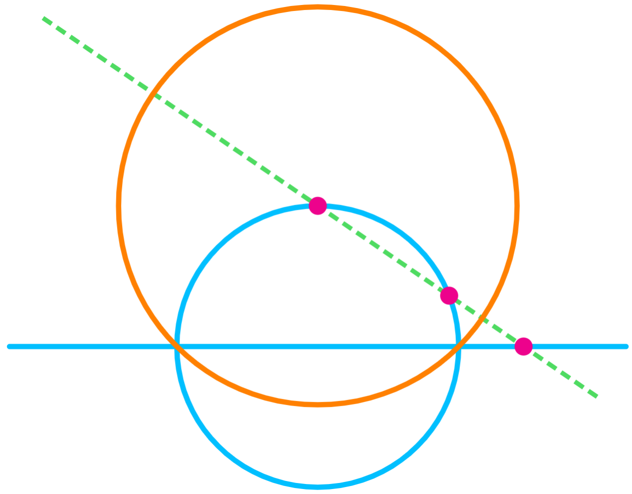
To turn any (distinct) pair of points A and B into any other pair C and D:

- First, apply an expansion or compression (from any central point), using two concentric circles, so the distance $\overline{AB}$ matches $\overline{CD}$.
- Second, apply a translation using two parallel lines, slide until $A = C$.
- Third, apply a rotation using two intersecting lines, to rotate around A (which is now also C) until $B = D$ too.

These three transformations (scale, translate, rotate) can be taken in pretty much any order to achieve the same effect, not just this particular order.

These dilations, translations, and rotations are special cases of **Möbius transformations**, which are complex-valued functions of the form $\frac{az+b}{cz+d}$.

A particular inversion of interest is **stereographic projection**:



Usually this projection is a one-to-one correspondence between points on a line and on a circle (not counting one point on the circle). It is set up by drawing lines through a point on the circle (called the pole) and recording the intersections with the circle and a line (which must be parallel to the circle's tangent line at the pole). This is always just the restriction of an inversion!

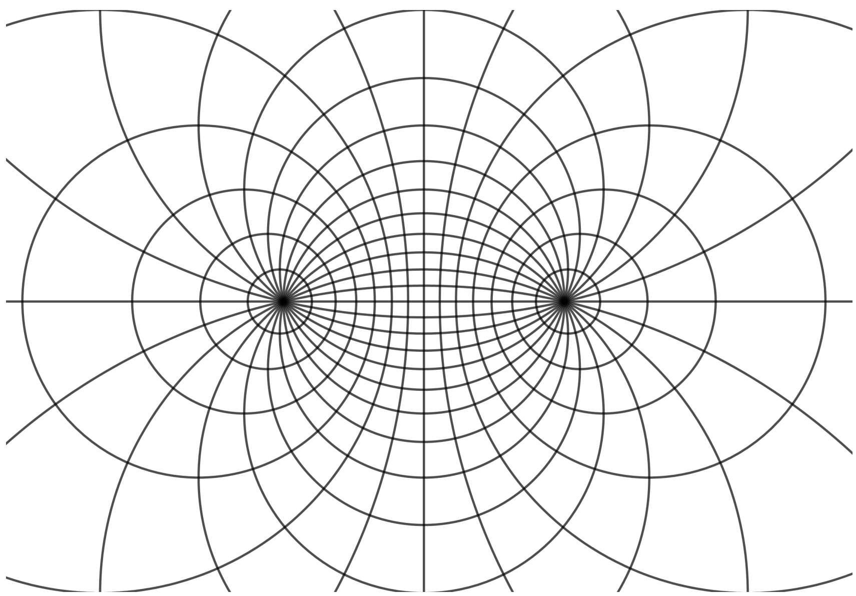
More generally, inversions preserve angles (they are **conformal**) and they also turn circles or lines into other circles or lines. Stereographically projecting the plane onto a sphere (itself a *spherical* inversion) gives us the perspective of the Riemann sphere - here, lines become circles on the sphere through the pole. It is from this perspective we see lines should be thought of as circles.

In astronomy this may be called the **celestial sphere**. The Möbius transformations (transported to the sphere from the plane) describe how the apparent night sky changes appearance if Earth were to travel through space at different speeds. For more serious changes to its appearance, speeds on the order of magnitude of the speed of light are necessary. The celestial sphere is the projectivized lightcone of special relativity, and Möbius transformations extend to linear Lorentz transformations of Minkowski spacetime.

The **Cartan-Dieudonnè theorem** for Euclidean space says every rotation is a composition of an even number of reflections across linear hyperplanes; here we've encountered a generalization of this theorem from Euclidean space to Minkowski spacetime: every Möbius transformation is a composition of an even number of inversions.

A group of transformations acting on a set of points is called **transitive** if it is possible to transform any point into any other point; we have shown the group of Möbius transformations is *doubly* transitive. In fact, it is *sharply* 3-transitive: given any three distinct points a, b, c and any other three distinct points u, v, w on the Riemann sphere, there is one and only one Möbius transformation f for which $f(a) = u, f(b) = v, f(c) = w$.

What about composing a pair of inversions across other combinations of generalized circles? (Generalized circle is an umbrella term for both circles and lines.) If the generalized circlces intersect, the composition of inversions is a hyperbolic rotation around the two poles, whereas if they don't then the two poles become a repelling source and an attracting sink for moving points.



Notice all of the inversion compositions show a *doubling* effect: the distance, angle, scale factor, or appropriate bipolar coordinate "between" two lines or circles is doubled to give the effect of the composition of inversions. This is a manifestation of **spin**, a mysterious concept from physics once described by a mathematician as the "square root of geometry."