Buckminsterfullerene: Solution

Suppose a polyhedron has P pentagonal and H hexagonal faces. Let V, E, F denote the numbers of vertices, edges and faces it has.

Equation One. Euler's formula says that for convex polyhedra,

$$V - E + F = 2. \tag{1}$$

Equation Two. Every pentagonal face has 5 edges, and every hexagonal face has 6 edges, for a total of $5P + 6H$ edges. However, this double-counts the edges, since there are two faces on either side of each edge, so

$$E = \tfrac{1}{2}(5P + 6H). \tag{2}$$

Equation Three. Each face is either a pentagon or hexagon, so

$$F = P + H. \tag{3}$$

Note equations $(1), (2), (3)$ allow us to solve for V, E, F:

$$\begin{cases} F = P + H, \\ E = \tfrac{5}{2}P + 3H, \\ V = \tfrac{3}{2}P + 2H + 2. \end{cases}$$

(Substitute (2) and (3) into (1) to solve for V above.)

Equation Four. Finally, let S be how many space diagonals there are.

The number of line segments joining distinct vertices is the combination ${}_VC_2$, called "V choose 2," also denoted $\binom{V}{2}$. These segments come in three classes: edges, face diagonals, and space diagonals. Counting by hand, there are 5 face diagonals per pentagon and 9 per hexagon, so $5P + 9H$ face diagonals.

So the fourth equation, from which we can solve for S by substituting, is

$$\binom{V}{2} = E + (5P + 9H) + S. \tag{4}$$