Anharmonic Asymmetry: Solution

In **cycle notation**, $(a_1 a_2 \cdots a_k)$ represents the permutation which cycles the elements $a_1, \cdots, a_k$ of a set (in that order). For example, for a set $\{1, 2, 3, 4\}$, the permutation (123) is the function ρ defined by the input/output pairs

$$\rho(1) = 2, \quad \rho(2) = 3, \quad \rho(3) = 1, \quad \rho(4) = 4.$$

(Any element x not listed in the cycle notation is fixed, i.e. $f(x) = x$.)

There are six permutations of $\{1, 2, 3\}$, listed below with probabilities:

(1)	(23)	(31)	(12)	(321)	(123)
Z	A	B	C	X	Y

When drawing a permutation from S_3, the probability it sends $i \mapsto j$ is the sum of the probabilities associated with every permutation that sends $i \mapsto j$. This gives us a 3×3 table of probabilities:

$\nearrow$	1	2	3
1	$A + Z$	$C + Y$	$B + X$
2	$C + X$	$B + Z$	$A + Y$
3	$B + Y$	$A + X$	$C + Z$

The ij entry is the probability of $i \mapsto j$. Since there is a 100% chance 1 is sent to one of $1, 2, 3$ we may conclude the first row sums to 1; similarly there is a 100% chance one of $1, 2, 3$ is sent to 1 so the first column also sums to 1. The same is true for all rows and columns, so the table is a so-called **doubly stochastic matrix**. In particular, all its entries ought to be 1/3.

Setting the entries equal to each other results in many of the variables being equal. For instance $A + X = A + Y = A + Z$ implies $X = Y = Z$ and $A + X = B + X = C + X$ implies $A = B = C$. Conversely, so long as A, B, C are equal and X, Y, Z are equal, the table's entries are all the same.

Thus, to get a nonuniform distribution on S_3 for which $i \mapsto j$ is equally likely for all pairs i, j it suffices to pick any solution of $A + X = \frac{1}{3}$ for which $A, X \geq 0$ and $A \neq X$, so such a distribution is possible.