Alfred's Ansatz: Solution

Rewrite $2^{62}+1$ as $4x^4+1$ with $x=2^{15}$. The polynomial $4x^4+1$ has no real root, since x^4 is can't be negative, so it cannot have a linear factor, and thus instead must have only quadratic factors (if it factors at all). Indeed, it must be a product of two quadratics. Their leading terms are either both $2x^2$ or else one is x^2 and the other is $4x^2$. Let's look at the first case and if that doesn't pan out explore the second. Write out the factorization

$$4x^4+1=(2x^2+ax+b)(2x^2+cx+d).$$

When expanded out and like terms collected, the right-hand side becomes

$$4x^4+2(a+c)x^3+(2b+2d+ac)x^2+(ad+bc)x+bd.$$

The coefficient of x^3 must be 0, so $c=-a$. The constant coefficient must be 1, so b and d are either both 1 or both -1 (in particular, $b=d$). We can now substitute $-a$ for c and b for d so there are only two unknowns. The coefficient of x becomes $ab-ba$ which is automatically 0. The coefficient of x^2 is now $4b-a^2$, which must be 0, so $a^2=4b$. This forces b to be nonnegative, so $b=1$ and $a=\pm 2$. Putting this all together we get the factorizations

$$4x^4+1=(2x^2+2x+1)(2x^2-2x+1),$$

$$2^{62}+1=(2^{31}+2^{16}+1)(2^{31}-2^{16}+1).$$

Each of the two factors above would work.

In general, the polynomial x^n-1 (and by extension, x^n+1) can be factored into the irreducible *cyclotomic polynomials* $\Phi_n(x)$. These can be "factored" further in a different sort of way as $\Phi(x)=U(x)^2\pm\bigcirc V(x)^2$ for polynomials $U(x),V(x)$ and monomial $\bigcirc$ depending on n (due to Lucas, Gauss, Schinzel).

This allows us to factor $\Phi_n(x)$ as an integer for particular values of x, which Aurifeuille (pseudonym Alfred de Caston) did in the case of

$$\begin{aligned}2^{2(2n+1)}+1&=(2^{2n+1}+1)^2-(2^{n+1})^2\\&=(2^{2n+1}+2^{n+1}+1)(2^{2n+1}-2^{n+1}+1).\end{aligned}$$

for $n=14$ (our case being $n=15$).