Problem of the week #6: Solution

For comparison, recall the dot product satisfies $\mathbf{N}_1 \cdot \mathbf{N}_2 = \|\mathbf{N}_1\|\|\mathbf{N}_2\| \cos\theta$, where θ is the angle between two vectors, so $\cos\theta$ is in terms of dot products:

$$\cos\theta = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{\sqrt{(\mathbf{N}_1 \cdot \mathbf{N}_1)(\mathbf{N}_2 \cdot \mathbf{N}_2)}}$$

In this case, suppose the two planes Π_1 and Π_2 (spanned by pairs $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}, \mathbf{D}$ respectively) intersect in a line perpendicular to a third plane Π. The so-called **dihedral angle** between Π_1 and Π_2 is ϕ (chosen to not be obtuse).

In the 2D plane Π we can see the two normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ of the first two planes Π_1 and Π_2 intersecting at an angle of either ϕ or else its supplement, depending on which normal vectors are chosen. This choice will only affect the sign of $\cos\phi$, so we may as well use any choice of normal vectors and take the absolute value.

To this end, normalize the cross products $\mathbf{A} \times \mathbf{B}$, $\mathbf{C} \times \mathbf{D}$ for $\mathbf{N}_1$, $\mathbf{N}_2$ in the relation $\cos\phi = \mathbf{N}_1 \cdot \mathbf{N}_2$, and also use the Binet-Cauchy identity:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) \quad = \quad (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

and special case $\|\mathbf{A} \times \mathbf{B}\|^2 = \|\mathbf{A}\|^2\|\mathbf{B}\|^2 - (\mathbf{A} \cdot \mathbf{B})^2$ (Lagrange's Identity).

This gives us the answer

$$\left| \frac{\mathbf{A} \times \mathbf{B}}{\|\mathbf{A} \times \mathbf{B}\|} \cdot \frac{\mathbf{C} \times \mathbf{D}}{\|\mathbf{C} \times \mathbf{D}\|} \right| = \frac{|(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})|}{\sqrt{\|\mathbf{A} \times \mathbf{B}\|^2 \|\mathbf{C} \times \mathbf{D}\|^2}}$$

$$= \quad \frac{|(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})|}{\sqrt{\big((\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})^2\big)\big((\mathbf{C} \cdot \mathbf{C})(\mathbf{D} \cdot \mathbf{D}) - (\mathbf{C} \cdot \mathbf{D})^2\big)}}.$$

All ten possible dot products of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ make an appearance.