Problem of the week #5: Solution

Since e^{-Cy^2} is constant with respect to x, the double integral is

$$\int_{-\infty}^{\infty} e^{-Cy^2}\left(\int_{-\infty}^{\infty} e^{-Ax^2-Bxy}\,\mathrm{d}x\right)\mathrm{d}y.$$

Complete the square in the inner integrand to get

$$\int_{-\infty}^{\infty} e^{-Cy^2+B^2/4A\,y^2}\left(\int_{-\infty}^{\infty} e^{-A(x+B/2A\,y)^2}\mathrm{d}x\right)\mathrm{d}y.$$

With the substitution $u = x + B/A\,y$ this becomes

$$\int_{-\infty}^{\infty} e^{[-C+B^2/4A]y^2}\left(\int_{-\infty}^{\infty} e^{-Au^2}\,\mathrm{d}u\right)\mathrm{d}y$$

Now u and y are independent, so the double integral factors as a product of integrals. Notice $A > 0$ and $C - B^2/4A > 0$ are necessary for the integrals to converge. Use the substitutions $[-C + B^2/4A]y^2 = -v^2$ and $-Au^2 = -w^2$:

$$\left(\int_{-\infty}^{\infty} e^{-v^2}\frac{\mathrm{d}v}{\sqrt{C-B^2/4A}}\right)\left(\int_{-\infty}^{\infty} e^{-w^2}\frac{\mathrm{d}w}{\sqrt{A}}\right)$$

This is two constants times two Gaussian integrals:

$$\frac{\sqrt{\pi}}{\sqrt{C-B^2/4A}}\cdot\frac{\sqrt{\pi}}{\sqrt{A}} \quad = \quad \frac{2\pi}{\sqrt{4AC-B^2}}.$$

Notice if we instead used the exponent $-(Ax^2 + 2Bxy + Cy^2)$ it would be

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{S} \begin{bmatrix} x \\ y \end{bmatrix} = Ax^2 + 2Bxy + Cy^2$$

and the integral would be $\pi/\sqrt{\det S}$. This generalizes to higher dimensions.