Problem of the week #4: Solution

One the one hand, $S_n(X,Y) = q^{G(n)} X^{F_n} Y^{F_{n-1}}$. On the other,

$$S_n(X,Y) = S_{n-1}(X,Y) S_{n-2}(X,Y)$$

Replace each of $S_{n-1}(X,Y)$ and $S_{n-2}(X,Y)$ to get

$$q^{G(n-1)} X^{F_{n-1}} Y^{F_{n-2}} \ \cdot \ q^{G(n-2)} X^{F_{n-2}} Y^{F_{n-3}}.$$

which simplifies, since q commutes with X and Y, to

$$q^{G(n-1)+G(n-2)} X^{F_{n-1}} Y^{F_{n-2}} X^{F_{n-2}} Y^{F_{n-3}}.$$

To proceed further, we must simplify $Y^{F_{n-2}} X^{F_{n-2}}$.

To do this, recall $YX = qXY$, which means sliding a Y right of an X introduces a q. Thus, sliding r-many Ys past an X would introduce a q^r, and sliding r-many Ys past s-many Xs introduces q^{rs}. Hence it simplifies to

$$q^{G(n-1)+G(n-2)+F_{n-2}^2} X^{F_{n-1}+F_{n-2}} Y^{F_{n-2}+F_{n-3}}.$$

Since this equals $q^{G(n)} X^{F_n} Y^{F_{n-1}}$, equating powers of q and subtracting:

$$G(n) - G(n-1) - G(n-2) = F_{n-2}^2.$$