Problem of the week #3: Solution

A nontrivial linear equation like the following is not possible:

$$Ap_{12} + Bp_{13} + Cp_{14} + Dp_{23} + Ep_{34} + Fp_{24} = 0.$$

Such an equation cannot exist because each term $x_i y_j$ appears no more than once, hence these terms cannot cancel in pairs. Thus, if an equation exists it must contain a product $p_{ij}p_{k\ell}$. When i, k, k, ℓ are all four numbers $1, 2, 3, 4$ we get the following three expressions:

$$\begin{aligned}
p_{12}p_{34} &= (x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) \\
&= x_1y_2x_3y_4 - x_1y_2y_3x_4 - y_1x_2x_3y_4 + y_1x_2y_3x_4 \\
p_{13}p_{42} &= (x_1y_3 - x_3y_1)(x_4y_2 - x_2y_4) \\
&= x_1y_2y_3x_4 - x_1x_2y_3y_4 - y_1y_2x_3x_4 + y_1x_2x_3y_4 \\
p_{14}p_{23} &= (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) \\
&= x_1x_2y_3y_4 - x_1y_2x_3y_4 - y_1x_2y_3x_4 + y_1y_2x_3x_4
\end{aligned}$$

The terms above have been written so the p-products' indices are even permutations of $1, 2, 3, 4$ and the x, y-products indices are simply $1, 2, 3, 4$ in order.

There are six possible "words" of two xs and two ys:

$$xxyy, \quad xyxy, \quad xyyx, \quad yxxy, \quad yxyx, \quad yyxx.$$

Each such term appears twice in the three p-products, once with a plus sign and once with a minus sign, hence their sum is zero:

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

(Or, equivalently, $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.)

Note this polynomial in the ps is half the determinant

$$\det \begin{bmatrix} x_1 & y_1 & x_1 & y_1 \\ x_2 & y_2 & x_2 & y_2 \\ x_3 & y_3 & x_3 & y_3 \\ x_4 & y_4 & x_4 & y_4 \end{bmatrix} = 2(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23})$$

This follows from expansion-by-minors: first pick one of the six minors p_{ij} in the first two columns, then a corresponding minor $p_{k\ell}$ in the last two columns; each product $p_{ij}p_{k\ell}$ is obtained in two ways. The determinant must vanish (i.e. equal zero) since the columns of the matrix are linearly dependent.

$\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ are known as **Plücker coordinates**.

Notice the p_{ij}s do not change under column operations (adding multiples of $\mathbf{x}$ to $\mathbf{y}$ or vice-versa), hence $\mathbf{p}$ depends only on span$\{\mathbf{x}, \mathbf{y}\}$.

The set of 2D subspaces of four-dimensional Euclidean space corresponds in Plücker coordinates to a subset of $\mathbb{R}^6$ defined by a system of algebraic equations. In affine space this is known as the (oriented) **Grassmanian** $\widetilde{G}(4, 2)$, or in projective space it is known as the **Klein quadraic**.