Problem of the week #2: Solution

The **tetration** operation $a \uparrow\uparrow b$ can initially be interpreted as repeated exponentiation, similar to how exponentiation can be interpreted as repeated multiplication and multiplication as repeated addition. (This kindergarten interpretation fails beyond counting numbers, of course.)

Also known as a "power tower," it is defined by the formula:

$$a \uparrow\uparrow b := \underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{b}.$$

E.g. $2 \uparrow\uparrow 3 := 2 \wedge 2 \wedge 2 = 16$. It satisfies the recurrence

$$a \uparrow\uparrow b = a \wedge (a \uparrow\uparrow (b-1)).$$

A pattern emerges applying iterated logarithms to tetrations, e.g.

$$\ln^3(a \uparrow\uparrow 5) = \ln\ln\ln a^{a^{a^{a^{a}}}} = a^a \ln(a\ln(a\ln a))$$

Notice $a^a = a \uparrow\uparrow 2$ and $\ln(a\ln(a\ln a)) = \ln^3(a \uparrow\uparrow 3)$. More generally,

$$\ln^c(a \uparrow\uparrow b) = (a \uparrow\uparrow (b-c)) \ln^c(a \uparrow\uparrow c)$$

when $c \le b$ (and $a \uparrow\uparrow 0 = 1$) by induction. So define

$$a_n := n \uparrow\uparrow (n+1), \qquad b_n := n \uparrow\uparrow n.$$

Then for $n > k$ (with k fixed) the difference $\ln^k a_n - \ln^k b_n$ is

$$\big[n \uparrow\uparrow (n+1-k) - n \uparrow\uparrow (n-k)\big] \ln^k(n \uparrow\uparrow k).$$

Both parts of the above product diverge as $n \to \infty$. The first may be written as $n^m - m$ where $m = n \uparrow\uparrow (n-k)$. Since $n^m \ge nm$ when $n > 1, m \ge 1$, which can be proved by induction (for fixed n), the bracketed expression is $\ge (n-1)m = (n-1)(n \uparrow\uparrow (n-k))$. So the bracketed expression diverges. For the non-bracketed expression,

$$\ln^k(n \uparrow\uparrow k) = \ln n + \ln\ln n + \ln\ln\ln n + \cdots + \ln^k n$$

by induction, which diverges since it is a sum of k divergent terms.