Problem of the week #1: Solution

Let $Q = V^2/S^3$ by the isoepiareal ratio. Suppose a cuboid with maximal Q has dimensions x, y, z so its volume and surface area are

$$V = xyz, \qquad S = 2xy + 2yz + 2zx.$$

Solution 1. If Q is maximized, then so is $\ln Q$, which is given by

$$\ln Q = 2\ln(xyz) - 3\ln(xy + yz + zx) - 3\ln 2.$$

If two of x, y, z are fixed while the third is changed, the ratio is decreased, so $\ln Q$ has a local maximum and its first derivative (with respect to the nonconstant variable) must vanish. Thus,

$$\begin{aligned}
\frac{\partial \ln Q}{\partial x} &= \frac{2}{x} - \frac{3(y+z)}{xy+yz+zx} &= 0 \\
\frac{\partial \ln Q}{\partial y} &= \frac{2}{y} - \frac{3(x+z)}{xy+yz+zx} &= 0 \\
\frac{\partial \ln Q}{\partial z} &= \frac{2}{z} - \frac{3(x+y)}{xy+yz+zx} &= 0
\end{aligned}$$

Solve for $\frac{2}{3}(xy + yz + zx)$ in each, equate the results:

$$x(y + z) = y(x + z) = z(x + y).$$

Subtracting pairs of expressions from this equation gives

$$0 = (x - y)z = (y - z)x = (x - z)y.$$

Since $x, y, z > 0$, the differences (e.g. $x - y$) are zero, so $x = y = z$.

This means the cuboid must be a cube.

Solution 2. Let a, b, c be the products xy, yz, zx. Then $V^2 = abc$ and $S = 2(a + b + c)$. With surface area S held constant, the AM-GM inequality implies the volume is bounded above by

$$\sqrt[3]{abc} \le \frac{a+b+c}{3} \implies V^{2/3} \le \frac{S}{6} \implies \frac{V^2}{S^3} \le \frac{1}{6^3}$$

with equality (hence when the ratio is maximized), crucially, if and only if $a = b = c$, which in turn is equivalent to $x = y = z$.