# Solution to Problem $\diamondsuit$–8

**Problem:** *For which values of a parameter $a$ the equation*

$$(\circledast) \qquad (a^2+1)(x^2+y^2) - 2(a+1)(x+y) + 2(1+2axy) = 0$$

*has exactly one solution $(x,y) \in \mathbb{R}^2$ ?*

*Solution.* Equation $(\circledast)$ can be transformed equivalently to
$(a^2x^2 + 2axy + y^2) + (a^2y^2 + 2axy + x^2) - 2(ax+y) - 2(ay+x) + 2 = 0$
and then to

$$[(ax+y)^2 - 2(ax+y) + 1] + [(ay+x)^2 - 2(ay+x) + 1] = 0.$$

Consequently, equation $(\circledast)$ is equivalent to

$$(ax+y-1)^2 + (ay+x-1)^2 = 0.$$

Sum of two squares of real numbers is zero if and only if both numbers are 0. Therefore, Equation $(\circledast)$ is equivalent to the following system of two linear equations:

$$ax + y = 1 \text{ and } x + ay = 1$$

If $a = 0$ then the system has exactly one solution $x = y = 1$. Otherwise, the linear equations represent lines with slopes $-\frac{1}{a}$ and $-a$. They intersect at exactly one point if and only if they are not parallel, i.e., they have different slopes. Hence, equation $(\circledast)$ has exactly one solution if and only if $a \notin \{-1, 1\}$. $\square$

Correct solution was received from :

(1) Grant Moles — POW 8: $\diamondsuit$