Solution to Problem ♦-8

Problem: For which values of a parameter $a$ the equation

\[(\star) \quad (a^2 + 1)(x^2 + y^2) - 2(a + 1)(x + y) + 2(1 + 2axy) = 0\]

has exactly one solution $(x, y) \in \mathbb{R}^2$?

Solution. Equation $(\star)$ can be transformed equivalently to

\[\left( a^2 x^2 + 2axy + y^2 \right) + \left( a^2 y^2 + 2axy + x^2 \right) - 2(ax + y) - 2(ay + x) + 2 = 0\]

and then to

\[\left[ (ax + y)^2 - 2(ax + y) + 1 \right] + \left[ (ay + x)^2 - 2(ay + x) + 1 \right] = 0.\]

Consequently, equation $(\star)$ is equivalent to

\[(ax + y - 1)^2 + (ay + x - 1)^2 = 0.\]

Sum of two squares of real numbers is zero if and only if both numbers are 0. Therefore, Equation $(\star)$ is equivalent to the following system of two linear equations:

\[ax + y = 1 \text{ and } x + ay = 1\]

If $a = 0$ then the system has exactly one solution $x = y = 1$. Otherwise, the linear equations represent lines with slopes $-\frac{1}{a}$ and $-a$. They intersect at exactly one point if and only if they are not parallel, i.e., they have different slopes. Hence, equation $(\star)$ has exactly one solution if and only if $a \not\in \{-1, 1\}$. □