Solution to Problem ♢–6

Problem: *Given*

$$y = \frac{\sqrt{1+2x}\cdot\sqrt[4]{1+4x}\cdot\sqrt[6]{1+6x}\cdot\ldots\cdot\sqrt[98]{1+98x}\,\sqrt[100]{1+100x}}{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+5x}\cdot\sqrt[7]{1+7x}\cdot\ldots\cdot\sqrt[99]{1+99x}\,\sqrt[101]{1+101x}}$$

find y' at $x = 0$.

Solution. For $n = 1, 2, 3, 4, \ldots, 50$ let

$$f_n(x) = \frac{\sqrt[2n]{1+2nx}}{\sqrt[2n+1]{1+(2n+1)x}}, \qquad x > -\frac{1}{200}.$$

Let

$$F(x) = \prod_{n=1}^{50} f_n(x) = \prod_{n=1}^{50} \frac{\sqrt[2n]{1+2nx}}{\sqrt[2n+1]{1+(2n+1)x}}, \qquad x > -\frac{1}{200}.$$

We want to find $F'(0)$. The functions f_n are differentiable on their domains $(-0.005, \infty)$ and

$$f_n'(x) = \left(\frac{(1+2nx)^{1/(2n)}}{(1+(2n+1)x)^{1/(2n+1)}}\right)' =$$

$$\frac{(1+2nx)^{(1-2n)/(2n)}(1+(2n+1)x)^{1/(2n+1)} - (1+2n)^{1/(2n)}(1+(2n+1)x)^{(-2n)/(2n+1)}}{(1+(2n+1)x)^{2/(2n+1)}}.$$

For $x = 0$ we get therefore $f_n'(0) = \frac{1-1}{1} = 0$ (for each $n = 1, \ldots, 50$). Since

$$F'(x) = \sum_{n=1}^{50}\left(f_n'(x)\cdot\prod_{1\le k\le 50,\ k\ne n} f_k(x)\right)$$

we conclude that $F'(0) = 0$. □

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