Solution to Problem $\diamondsuit$–5

Problem: *Find all complex numbers x, y, z which satisfy*

$$x + y + z = x^2 + y^2 + z^2 = x^3 + y^3 + z^3 = 3.$$

Solution. First we note that $1, 1, 1$ is a solution to our system of three equations.

Suppose that complex number x, y, z satisfy

(1) $x + y + z = 3$,
(2) $x^2 + y^2 + z^2 = 3$,
(3) $x^3 + y^3 + z^3 = 3$.

Since

(4) $(x + y + z)^2 = (x^2 + y^2 + z^2) + 2(xy + yz + zx)$,

we have $3^2 = 3 + 2(xy + yz + zx)$, so we conclude that

(5) $xy + yz + zx = 3$.

Since

$$(x+y+z)^3 = (x^3+y^3+z^3)+3(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+6xyz,$$

we have

(6) $8 = (x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 2xyz$.

Now,

(7) $(x + y + z)(xy + yz + zx) = (x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 3xyz$,

and hence

(8) $x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2 = 9 - 3xyz$.

Together with equation (6) this gives $8 = 9 - 3xyz + 2xyz$. Thus

(9) $xyz = 1$.

Vieta's formulas and equations (1), (5) and (9) imply that each of the numbers x, y, z is a root of the cubic

$$w^3 - 3w^2 + 3w - 1 = (w - 1)^3.$$

Consequently $x = y = z = 1$ $\square$

Correct solution was received from :

(1) Grant Moles POW 5: $\diamondsuit$