Solution to Problem ◊-3

Problem: Find the maximum height above the x-axis of the cardioid $r = 1 + \cos(\theta)$.

Solution. On the given cardioid,

$$x = (1 + \cos(\theta)) \cos(\theta) \quad \text{and} \quad y = (1 + \cos(\theta)) \sin(\theta).$$

The question is to find the maximum value $y_{\text{max}}$ of $y$. Note that $y > 0$ is equivalent to $\sin(\theta) > 0$. From

$$\frac{dy}{d\theta} = 2\cos^2(\theta) + \cos(\theta) - 1$$

we get that the critical numbers of the function $y = y(\theta)$ are the values of $\theta$ for which $\cos(\theta) = -1$ or $\cos(\theta) = \frac{1}{2}$. Since $y_{\text{max}} > 0$ (and thus $\sin(\theta) > 0$) it follows that $\theta_{\text{max}}$ giving the maximum value $y_{\text{max}}$ of $y$ satisfies

$$\sin(\theta_{\text{max}}) = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

and the maximum height equals

$$y_{\text{max}} = \left(1 + \frac{1}{2}\right) \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}.$$