Solution to Problem ♢–2

Problem: *For what values of the constant $c > 0$ does the equation $\ln(x) = cx^2$ have solutions?*

Solution. Assume $c > 0$ and let $f(x) = \ln(x) - cx^2$ for $x > 0$. Thus the domain of f is the interval $(0, \infty)$. Note that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to \infty} f(x) = -\infty.$$

From $f'(x) = \frac{1}{x} - 2cx$ it follows that there is a local (and absolute) maximum at $x = \frac{1}{\sqrt{2c}}$. Since the function f is continuous on its domain, by the Intermediate Value Theorem, it will have a root if and only if $f\left(\frac{1}{\sqrt{2c}}\right) \geq 0$. But

$$\ln\left(\frac{1}{\sqrt{2c}}\right) - c \cdot \frac{1}{2c} \geq 0 \quad \text{if and only if}$$

$$\ln\left(\frac{1}{\sqrt{2c}}\right) \geq \frac{1}{2} \quad \text{if and only if}$$

$$c \leq \frac{1}{2e}.$$

Consequently, if $c > 0$ then the equation $\ln(x) = cx^2$ has solutions if and only if $c \leq \frac{1}{2e}$. □

Correct solution was received from :

(1) Jacob Cleveland — POW 2: ♢
(2) Grant Moles — POW 2: ♢
(3) Zach Sabata — POW 2: ♢
(4) Brad Tuttle — POW 2: ♢