Solution to Problem $\diamondsuit$–12

Problem: *Let X be the set $\{1,2,\ldots,20\}$ and let P be the set of all 9–element subsets of X. Show that for every function $f : P \longrightarrow X$ we can find a 10–element subset Y of X, such that $f\big(Y \setminus \{k\}\big) \neq k$ for any $k \in Y$.*

Solution. Put

$$\mathcal{Y} = \{(S,k) \in P \times X : f(S) = k\}.$$

Evidently, the set $\mathcal{Y}$ has $\binom{20}{9}$ elements, since we can choose any $S \in P$ and k is then fixed. Now let

$$\mathcal{X} = \big\{(Y,k) : k \in Y \subseteq X \ \wedge \ |Y| = 10 \ \wedge \ f\big(Y \setminus \{k\}\big) = k\big\},$$

and let

$$Q = \big\{Y \subseteq X : \big(\exists k \in X\big)\big((Y,k) \in \mathcal{X}\big)\big\}.$$

Clearly $|Q| \leq |\mathcal{X}|$. The mapping

$$\pi : \mathcal{X} \longrightarrow \mathcal{Y} : (Y,k) \to (Y \setminus \{k\}, k)$$

is an injection because if $(Y \setminus \{k\}, k) = (Y' \setminus \{k'\}, k')$, then $k = k'$ and hence $Y = Y'$. Consequently,

$$\big|Q\big| \leq \big|\mathcal{X}\big| \leq \big|\mathcal{Y}\big| = \binom{20}{9}.$$

But there are $\binom{20}{10}$ subsets $Y \subseteq X$ with 10 elements, so at least $\binom{20}{10} - \binom{20}{9}$ of them (more than 16000) do not belong to Q, in other words they are such that $f\big(Y \setminus \{k\}\big) \neq k$ for any $k \in Y$. $\square$

Correct solution was received from :

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