Solution to Problem $\diamondsuit$–10

Problem: *How many solutions in real numbers does the equation*

$$(\heartsuit) \qquad (\sqrt{3}-1)^x = 2(\sqrt{2}+1)^x + 1$$

have?

Solution. First we know that the function

$$F(x) = (\sqrt{3}-1)^x - 2(\sqrt{2}+1)^x - 1, \qquad x \in \mathbb{R}$$

is differentiable on $\mathbb{R}$ and $F(0) = -2 < 0$. Also,

- $\lim\limits_{x\to-\infty} (\sqrt{3}-1)^x = \infty$ (since $\sqrt{3}-1 < 1$), and
- $\lim\limits_{x\to-\infty} (\sqrt{2}+1)^x = 0$ (since $\sqrt{2}+1 > 1$).

Consequently $\lim\limits_{x\to-\infty} F(x) = \infty$ and for sufficiently small x^* we will have $F(x^*) > 0$. Therefore, by the Intermediate Value Theorem, the function F takes value 0 for some $x_0 < 0$ and the equation $(\heartsuit)$ has *at least one solution.*

Now note that, for every $x \in \mathbb{R}$,

$$F'(x) = \ln(\sqrt{3}-1)\cdot(\sqrt{3}-1)^x - 2\ln(\sqrt{2}+1)\cdot(\sqrt{2}+1)^x.$$

and $\ln(\sqrt{3}-1) < 0$, $2\ln(\sqrt{2}+1) > 0$, $(\sqrt{3}-1)^x > 0$ and $(\sqrt{2}+1)^x > 0$. Therefore $F'(x) < 0$ for all $x \in \mathbb{R}$ and the function F is *stricly decreasing*. In particular, F is one-to-one. Consequently, there is *exactly one solution* to the equation $(\heartsuit)$. $\square$

CORRECT SOLUTION WAS RECEIVED FROM :

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