Solution to Problem $\diamondsuit$–1

Problem: *Let* $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *be the set of all natural numbers and let* $\mid$ *be the divisibility relation on* $\mathbb{N}$, *i.e.,*

$$m \mid n \quad \textit{if and only if} \quad \big(\exists k \in \mathbb{N}\big)\big(m \cdot k = n\big).$$

We say that a set $A \subseteq \mathbb{N}$ *is* a $\mid$–chain *if* $\big(\forall n, m \in A\big)\big(n \mid m \ \vee m \mid n\big)$, *and the set* A *will be called* an $\mid$–antichain *if* $\big(\forall n, m \in A\big)\big(n \mid m \ \Rightarrow \ m = n\big)$. *Suppose that we have finitely many* $\mid$*–chains* $A_1, A_2, \ldots, A_k \subseteq \mathbb{N}$ *and finitely many* $\mid$*–antichains* $B_1, B_2, \ldots, B_\ell \subseteq \mathbb{N}$. *Show that*

$$A_1 \cup A_2 \cup \ldots \cup A_k \cup B_1 \cup B_2 \cup \ldots \cup B_\ell \neq \mathbb{N}.$$

Solution. For $m \in \mathbb{N}$ let $\Psi(m)$ be the number of primes in the prime factorization counting with repetitions (so $\Psi(2^5 \cdot 3^6 \cdot 7^3) = 14$). Note that

- $(\circledast)_1$ if $n|m$, $n \neq m$, then $\Psi(n) < \Psi(m) < m$ and consequently
- $(\circledast)_2$ if $A \subseteq \mathbb{N}$ is a $\mid$–chain and $m \in A$, then $|\{a \in A : a < m\}| \leq \Psi(m)$.

Let $A_1, A_2, \ldots, A_k \subseteq \mathbb{N}$ be non-empty $\mid$–chains. Every finite $\mid$–chain can be extended to an infinite chain (e.g., by multiplying the largest element by powers of 2), we may assume that each A_i is infinite.

Also, let $B_1, B_2, \ldots, B_\ell \subseteq \mathbb{N}$ be non-empty $\mid$–antichains. For $i = 1, \ldots, \ell$ put $b_i = \min(B_i)$ and note that

- $(\circledast)_3$ no element of $B_i \setminus \{b_i\}$ is a multiple of b_i.

Choose a prime number p so large that for every $j = 1, \ldots, k$

- $(\circledast)_4$ $b_1 \cdot \ldots \cdot b_\ell + 2 < |\{a \in A_j : a < p\}|$.

Consider the number $N = p \cdot b_1 \cdot \ldots \cdot b_\ell$. Since N is a multiple of b_i (for each $i = 1, \ldots, \ell$), it follows from $(\circledast)_3$ that

- $(\heartsuit)_1$ $N \notin B_i$ for all $i = 1, \ldots, \ell$.

Now, fix $j \in \{1, \ldots, k\}$ and consider $M_j = \max\big(\{a \in A_j : a < N\}\big)$. It follows from $(\circledast)_4$ and $(\circledast)_2$ that

$$\Psi(N) \leq b_1 \cdot \ldots \cdot b_\ell + 1 < |\{a \in A_j : a < M_j\}| < \Psi(M_j).$$

By $(\circledast)_1$ we may conclude now that M_j does not divide N, and consequently $N \notin A_j$. Thus we have shown that

- $(\heartsuit)_2$ $N \notin A_j$ for all $j = 1, \ldots, k$.

Putting $(\heartsuit)_1$ and $(\heartsuit)_2$ together we get $N \notin A_1 \cup A_2 \cup \ldots \cup A_k \cup B_1 \cup B_2 \cup \ldots \cup B_\ell$. $\square$

CORRECT SOLUTION WAS RECEIVED FROM :

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