Solution to Problem ♦-1

Problem: Let \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \) be the set of all natural numbers and let \( | \) be the divisibility relation on \( \mathbb{N} \), i.e.,

\[
m \mid n \quad \text{if and only if} \quad (\exists k \in \mathbb{N})(m \cdot k = n).
\]

We say that a set \( A \subseteq \mathbb{N} \) is a \(|\)–chain if \((\forall n, m \in A)(n \mid m \lor m \mid n)\), and the set \( A \) will be called an \(|\)–antichain if \((\forall n, m \in A)(n \mid m \Rightarrow m = n)\). Suppose that we have finitely many \(|\)–chains \( A_1, A_2, \ldots, A_k \subseteq \mathbb{N} \) and finitely many \(|\)–antichains \( B_1, B_2, \ldots, B_\ell \subseteq \mathbb{N} \). Show that

\[
A_1 \cup A_2 \cup \ldots \cup A_k \cup B_1 \cup B_2 \cup \ldots \cup B_\ell \neq \mathbb{N}.
\]

Solution. For \( m \in \mathbb{N} \) let \( \Psi(m) \) be the number of primes in the prime factorization counting with repetitions (so \( \Psi(2^5 \cdot 3^6 \cdot 7^3) = 14 \)). Note that

\((\ast)_1\) if \( n \mid m, n \neq m \), then \( \Psi(n) < \Psi(m) < m \) and consequently

\((\ast)_2\) if \( A \subseteq \mathbb{N} \) is a \(|\)–chain and \( m \in A \), then \( |\{a \in A : a < m\}| \leq \Psi(m) \).

Let \( A_1, A_2, \ldots, A_k \subseteq \mathbb{N} \) be non-empty \(|\)–chains. Every finite \(|\)–chain can be extended to an infinite chain (e.g., by multiplying the largest element by powers of 2), we may assume that each \( A_i \) is infinite.

Also, let \( B_1, B_2, \ldots, B_\ell \subseteq \mathbb{N} \) be non-empty \(|\)–antichains. For \( i = 1, \ldots, \ell \) put \( b_i = \min(B_i) \) and note that

\((\ast)_3\) no element of \( B_i \setminus \{b_i\} \) is a multiple of \( b_i \).

Choose a prime number \( p \) so large that for every \( j = 1, \ldots, k \)

\((\ast)_4\) \( b_1 \cdot \ldots \cdot b_\ell + 2 < |\{a \in A_j : a < p\}|. \)

Consider the number \( N = p \cdot b_1 \cdot \ldots \cdot b_\ell \). Since \( N \) is a multiple of \( b_i \) (for each \( i = 1, \ldots, \ell \)), it follows from \((\ast)_3\) that

\((\bowtie)_1\) \( N \notin B_i \) for all \( i = 1, \ldots, \ell \).

Now, fix \( j \in \{1, \ldots, k\} \) and consider \( M_j = \max\{\{a \in A_j : a < N\}\} \). It follows from \((\ast)_4\) and \((\ast)_2\) that

\[
\Psi(N) \leq b_1 \cdot \ldots \cdot b_\ell + 1 < |\{a \in A_j : a < M_j\}| < \Psi(M_j).
\]

By \((\ast)_1\) we may conclude now that \( M_j \) does not divide \( N \), and consequently \( N \notin A_j \). Thus we have shown that

\((\bowtie)_2\) \( N \notin A_j \) for all \( j = 1, \ldots, k \).

Putting \((\bowtie)_1\) and \((\bowtie)_2\) together we get \( N \notin A_1 \cup A_2 \cup \ldots \cup A_k \cup B_1 \cup B_2 \cup \ldots \cup B_\ell \). \( \square \)

{**Correct solution was received from:**

1. Grant Moles

POW 1: ♦