Problem $\diamondsuit$–1

Due in DSC 222 by 12 noon, **Friday, January 18**, *2019*

Problem: Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ be the set of all natural numbers and let $|$ be the divisibility relation on $\mathbb{N}$, i.e.,

$$m \mid n \quad \text{if and only if} \quad \big(\exists k \in \mathbb{N}\big)\big(m \cdot k = n\big).$$

We say that a set $A \subseteq \mathbb{N}$ is *a* $|$*–chain* if

$$\big(\forall n, m \in A\big)\big(n \mid m \ \vee m \mid n\big),$$

and the set A will be called *an* $|$*–antichain* if

$$\big(\forall n, m \in A\big)\big(n \mid m \ \Rightarrow \ m = n\big).$$

Suppose that we have finitely many $|$–chains $A_1, A_2, \ldots, A_k \subseteq \mathbb{N}$ and finitely many $|$–antichains $B_1, B_2, \ldots, B_\ell \subseteq \mathbb{N}$. Show that

$$A_1 \cup A_2 \cup \ldots \cup A_k \cup B_1 \cup B_2 \cup \ldots \cup B_\ell \neq \mathbb{N}.$$

RULES:

- The competition is open to all *undergraduate* UNO students and it is supervised by *Upper Curriculum Committee* of the Mathematics Department.
- Submit your solutions to Andrzej Rosłanowski in DSC 222 or to his mailbox.
- Every nontrivial step/claim in your solution must justified. You may cite/quote a result from your textbook, past problems of the week and other widely available sources. In each case you have to give full reference.
- There are no partial credits, so rather err on the side of caution and provide more explanations than less. If you are not sure that your sources/references are appropriate, please include the complete relevant proofs from there.
- Your answers should be be written clearly and legibly. We reserve the right to refuse grading your work if it is difficult to read it.
- The winners of Spring 2019 edition of POW will be determined at the end of the semester based on the number of correct solutions submitted.
- Problems will be posted by Friday 5pm and the solutions are due by the following Friday 12 noon.

PRIZES:

- Winners will receive books published by the American Mathematical Society. The titles actually awarded will be selected in cooperation with the awardees.
- Everybody scoring in the POW Competition qualifies for the grand finale:

  $\frac{\pi}{2}$ *Mathematical Competition.*