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Solutions to Problems from the $\frac{\pi}{2}$ *Mathematical Contest*

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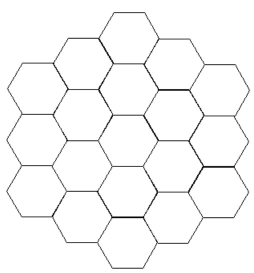
Friday, April 19, 2019

Problem 1: *In a square with sides of length 10 in the centers of adjacent sides were connected by line segments and resulting four triangle were cut off. What is the area of the remaining figure?*

Solution: After cutting off the corners we are left with a square. The four triangles that were cut off can be put together to create a congruent square. Therefore, the are of the remaining part is a half of the area of the original square, i.e.,

$$\frac{1}{2} \cdot 10 \cdot 10 = 50 \ \ [\text{in}^2].$$

Problem 2: *A honeycomb is a mass of hexagonal prismatic wax cells built by honey bees. The figure represents a honeycomb with "side" of length 3 cells. How many cells do a honeycomb with a "side" of 10 cells have ?*



Solution: The honeycomb described in th eproblem can be constructed as follows. We start with one cell, and around it we build a layer of 6 cells (creating a honecomb of side 2 cells). Then we add another (second) layer with 6+6 cells: 6 are added "between" any two adjacent cells from the previous layer, and 6 are added "on top" of each (corner) cell from the previous layer. Then we add the third layer of cells: one cell "between" any two adjacent cells and 6 cells "on top" of corner cells from the second layer. Thus the third layer will have 12+6 cells. The fourth layer is added to it and it will 18+6 cells (one cell "between" any two adjacent cells from the previous leyer and 6 cells "on top" of corner cells from the prevoius layer). The process continues until we set nineth layer of cells with 48+6 cells. Then the total number of cells will be

$$1 + 6 + 12 + 18 + 24 + 30 + 36 + 42 + 48 + 54 = \frac{10 \cdot 55}{2} = 271.$$

Problem 3: *Three boys, Adam, Bill and Charlie, ate 14 donuts. Adam ate twice less than Charlie and Bill ate more than Adam but less than Charlie. How many donuts did each of the boys eat?*

Solution: Let a, b, c be the numbers of donats eaten by Adam, Bill and Charlie, respectively. Then a, b, c are non-negative integers satisfying the following conditions:

- $a + b + c = 14$,
- $2a = c$, and
- $a < b < c$.

Consequently, also $b = 14 - 3a$.

If $a \geq 5$ then $14 - 3a < 0$ so these values of a are not possible. If $a \leq 2$ then $c \leq 4$ and it impossible to choose $b < c$ satisfying $a + b + c = 14$.

If $a = 4$ then $14 - 3a = 2 < a$, so this value of a is not possible.

If $a = 3$ then $a < 14 - 3a = 5 < 6 = 2a = c$. Thus $a = 3$, $b = 5$ and $c = 6$ are the only integers saisfying our conditions. Consequently, Adam ate 3 donuts, Bill ate 5 donuts and Charlie ate 6 donuts.

Problem 4: *In the Euclidean plane somebody marked 100 horizontal lines and 2019 vertical lines. How many rectangles with sides included in these lines are there ?*

Solution: Each such rectangle is determined by choosing two verital lines and two horizontal line. The first choice can be made in $\frac{2019 \cdot 2018}{2}$ ways and the second selection in $\frac{100 \cdot 99}{2}$ ways. Since these to selections are independent, the total number of the resulting rectangles is

$$\frac{2019 \cdot 2018}{2} \cdot \frac{100 \cdot 99}{2} = 2019 \cdot 1009 \cdot 50 \cdot 99 = 10083996450.$$

Problem 5: *A triangle ABC has area 10* in^2*. A point P lies on the side AC and satisfies*

$$\frac{|AP|}{|PC|} = 4.$$

Find the area of the triangle BCP.

Solution: Let h be the height of the triangle ABC perpendicular to base AC. Then h is also the height of the triangle BCP perpendicular to base PC. Since

$$|AC| = |AP| + |PC| = 4 \cdot |PC| + |PC| = 5 \cdot |PC|$$

we have

$$10 = \text{Area}(\triangle ABC) = \frac{1}{2} \cdot h \cdot |AC| = \frac{1}{2} \cdot h \cdot 5 \cdot |PC| = 5 \cdot \text{Area}(\triangle BCP).$$

Hence, $\text{Area}(\triangle BCP) = 2\ \text{in}^2$.

Problem 6: *You try to build a rectangular cuboid (a.k.a. rectangular parallelepiped) using 105 unit cubes stuck together so that neighboring cubes touch each other with full facets. How many non-congruent cuboids can you construct this way?*

Solution: First note that $105 = 3 \cdot 5 \cdot 7$. Therefore possible dimensions of our rectangular cuboids are

- $1 \times 1 \times 105$,
- $1 \times 3 \times 35$,
- $1 \times 5 \times 21$,
- $1 \times 7 \times 15$, and
- $3 \times 5 \times 7$.

Thus there are 5 non-congruent cuboids satisfying our demands.

Problem 7: *Show that for any positive reals x, y, z we have*

$$x^x y^y z^z \geq (xyz)^a,$$

where a is the arithmetic mean of x, y, z.

Solution: Without loss of generality $x \geq y \geq z$. We have

$$x^x y^y \geq x^y y^x,$$

because that is equivalent to

$$(x/y)^x \geq (x/y)^y$$

which is obviously true. Similarly,

$$y^y z^z \geq y^z z^y \qquad \text{and} \qquad z^z x^x \geq z^x x^z.$$

Multiplying these three together we get

$$(x^x y^y z^z)^x \geq x^{y+z} y^{z+x} z^{x+y}.$$

Multiplying both sides by $x^x y^y z^z$ gives

$$(x^x y^y z^z)^3 \geq (xyz)^{3a}.$$

Taking cube roots gives the required result.

Problem 8. *Suppose that p, q are distinct prime numbers and m, n are natural numbers and*

$$mp \equiv 1 \mod q \qquad \textit{and} \qquad nq \equiv 1 \mod p.$$

Show that $mp + nq > pq$.

Solution: By our assumptions we may find natural numbers x and y such that

$$mp = qx + 1 \quad \text{and} \quad nq = py + 1.$$

Then

$$m + y = \frac{qx+1}{p} + \frac{nq-1}{p} = \frac{q(x+n)}{p}.$$

Since p and q are distinct prime numbers and $m + y$ is a natural number, p must divide $x + n$. Consequently $1 \leq \frac{x+n}{p}$ and hence $q \leq \frac{q(x+n)}{p} = m + y$. Therefore

$$mp + nq = mp + py + 1 = p(m + y) + 1 \geq pq + 1 > pq,$$

as desired.

Problem 9: *Find all integer solutions of the equation*

$$(\heartsuit) \qquad x^3 + 2y^3 + 4z^3 = 6xyz.$$

Solution: We note that if a triple (x, y, z) satisfies the equation $(\heartsuit)$, then also the triple $(x/2, y/2, z/2)$ satisfies it, and also $(x/4, y/4, z/4)$ is a solution to $(\heartsuit)$ etc. Thus

$(*)_1$ if a triple (x, y, z) satisfies the equation $(\heartsuit)$ and $n \in \mathbb{N}$, then also the triple $(x/2^n, y/2^n, z/2^n)$ satisfies $(\heartsuit)$.

Now, suppose (x, y, z) is an integer solution of the equation $(\heartsuit)$. Then $x^3 = 6xyz - 2y^3 - 4z^3$, so x is an even number. Similarly y and z must be even. By $(*)_1$, $(x/2, y/2, z/2)$ is an integer solution of the equation $(\heartsuit)$. Consequently,

$(*)_2$ if (x, y, z) an integer solution of the equation $(\heartsuit)$ and $n \in \mathbb{N}$, then also the triple $(x/2^n, y/2^n, z/2^n)$ an integer solution of the equation $(\heartsuit)$.

It follows from $(*)_2$ that if (x, y, z) an integer solution of the equation $(\heartsuit)$, then $x = 0$, $y = 0$ and $z = 0$.

We easily verify that $x = y = z = 0$ satisfy $(\heartsuit)$, so this is the only integer solution of this equation.

Problem 10: *Show that for each natural number n we can find an n–digit integer with all its digits odd which is divisible by 5^n.*

Solution: We show this claim by induction on n. Let $\Phi(n)$ be the assertion that

there exists an n–digit integer with all its digits odd which is divisible by 5^n.

We will verify that the formula $\Phi(n)$ satisfies the assumptions of the Theorem on Mathematical Induction.

Basic Step: For $n = 1$ the number 5 witnesses that $\Phi(1)$ is true.
Inductive Step: Suppose that $\Phi(n)$ holds true and let an n–digit number N witness this. Consider the five $n + 1$ digit numbers

$$10^n + N, \quad 3 \cdot 10^n + N, \quad 5 \cdot 10^n + N, \quad 7 \cdot 10^n + N, \quad \text{and} \quad 9 \cdot 10^n + N.$$

We may take out the common factor 5^n to get the five numbers

$$k, \quad k + 2^{n+1}, \quad k + 2 \cdot 2^{n+1}, \quad k + 3 \cdot 2^{n+1}, \quad \text{and} \quad k + 4 \cdot 2^{n+1},$$

for some k. Since 2^{n+1} is relatively prime to 5, the five numbers are all incongruent modulo 5 and so one must be a multiple of 5. Consequently, the corresponding number $x \cdot 10^n + N$ will winess $\Phi(n + 1)$.

Thus Φ satisfies the assumptions of the Theorem on Mathematical Induction and therefore for every n the assertion $\Phi(n)$ holds true, as desired.