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Solutions to Problems from the $\frac{\pi}{2}$ *Mathematical Contest*

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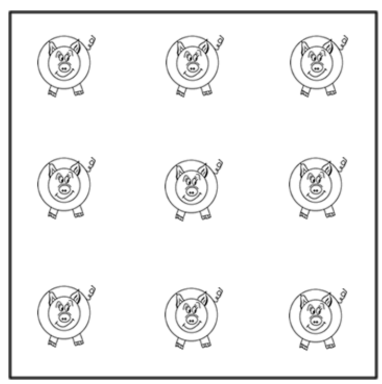
Saturday, April 21, 2018

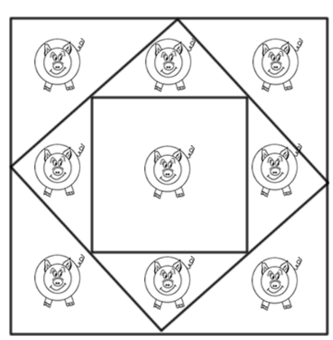
Problem 1: *A geometric progression $a_1, a_2, a_3, \ldots, a_n, \ldots$ satisfies $a_2 = 5$ and $a_3 = 1$. Find a_1.*

Solution: It follows from $\frac{a_2}{a_1} = \frac{a_3}{a_2}$ that $a_1 = \frac{a_2^2}{a_3} = 25$.

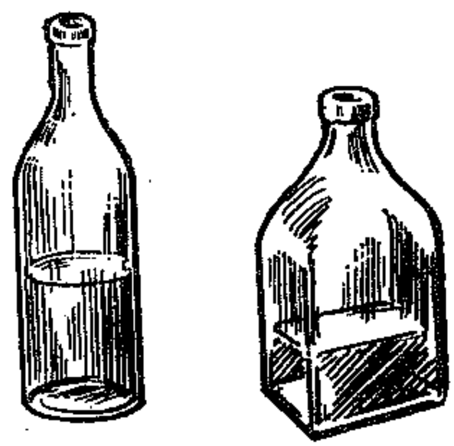
Problem 2: *Construct two squares that provide each pig with his own pen space.*



Solution:



Problem 3: *If a bottle, partly filled with liquid (like the pictures shown below), has a round, square, or rectangular bottom which is flat, can you find the volume of the bottle using only a ruler? You may not add or pour out liquid.*

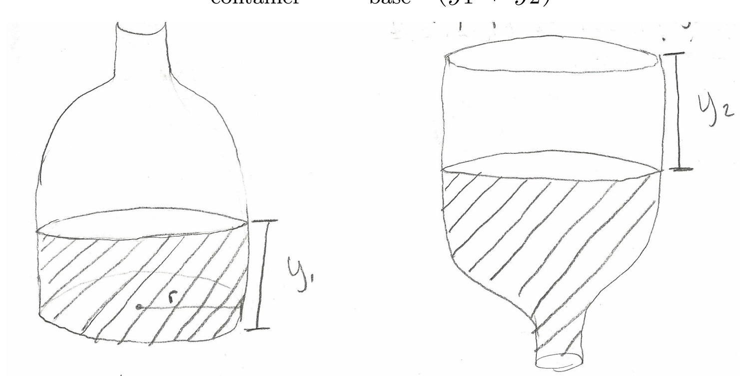


Solution: Calculate the are of the base, measuring the radius or the length of the sides with a ruler. Call this A_{base}; so $A_{\text{base}} = \pi r^2$ of $A_{\text{base}} = w \cdot \ell$.

With the ruler measure the depth of the liquid. Call this y_1. Then invert the bottle and measure the depth of the empty space. Call this y_2.

With these measurements,

$$V_{\text{container}} = A_{\text{base}} \cdot (y_1 + y_2).$$



Problem 4: *Find the integral*

$$\int_0^{\pi/2} e^{2x} \cos 2x \; dx$$

Solution: Using integration by parts we find that

$\int_0^{\pi/2} e^{2x} \cos 2x dx = \frac{1}{2} e^{2x} \cos 2x |_0^{\pi/2} + . \int_0^{\pi/2} e^{2x} \sin 2x dx = -\frac{1}{2} e^{\pi} - \frac{1}{2} - . \int_0^{\pi/2} e^{2x} \cos 2x dx$

Hence

$$\int_0^{\pi/2} e^{2x} \cos 2x dx = -\frac{1}{4}(e^{\pi} + 1).$$

Problem 5: *How many 9-digit numbers divisible by 5 could be obtained by permutations from the number* 377353752.

Solution: Because the numbers are divisible by 5, last digit has to be equal to 5, The number of distinct permutations of the first 8 digits is equal to multinomial coefficient $\binom{8}{3,3,1,1} = \frac{8!}{3!3!} = 1120$.

Problem 6: *Prove that for each natural number* $n \geq 3$ *we have*

$$(n+1)^n < n^{n+1}.$$

Solution: The inequality $(n+1)^n < n^{n+1}$ is equivalent to $\frac{(n+1)^n}{n^n} < n$, or $\left(1 + \frac{1}{n}\right)^n < n$. We will show the latter inequality by induction on $n \geq 3$

Basic step: For $n = 3$ we have

$$\left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} < \frac{81}{27} = 3.$$

Inductive step: Let us assume

$(\otimes)_n \; \left(1 + \frac{1}{n}\right)^n < n.$

We are going to argue that $(\otimes)_{n+1}$ holds true then. For this we just note that

$$\left(1+\frac{1}{n+1}\right)^{n+1} < \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n} \cdot \left(1+\frac{1}{n}\right) <_{\text{by } (\otimes)_n} n \cdot \left(1+\frac{1}{n}\right) = n+1.$$

Problem 7: *A solid is generated by rotating about the x–axis the region under the curve $y = f(x)$, where f is a positive function and $x \geq 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 forall $b > 0$. Find the function f.*

Solution: The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi[f(x)]^2\,dx$. Hence,we are given that

$$\int_0^b \pi[f(x)]^2\,dx = b^2$$

for all $b > 0$. Differentiating both sides of this equation using the Fundamental Theorem of Calculus gives

$$2b = \pi[f(b)]^2.$$

Hence, since $f(x) > 0$,

$$f(x) = \sqrt{2x/\pi} \qquad \text{for all } x > 0.$$

Problem 8. *Find the sum of the roots of the equation*

$$x^2 - 31x + 220 = 2^x(31 - 2x - 2^x)$$

Solution: The equation $x^2 - 31x + 220 = 2^x(31 - 2x - 2^x)$ is equivalent to the equation

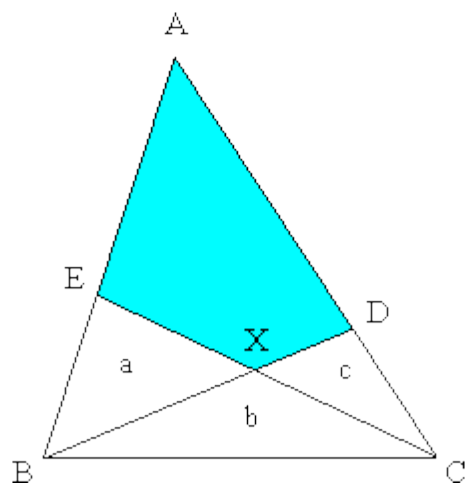
$$(x + 2^x)^2 - 31(x + 2^x) + 220 = 0.$$

Consequently, the roots of our original equation satisfy

$$x + 2^x = 11 \quad \text{or} \quad x + 2^x = 20.$$

Since the function $f(x) = x + 2^x$, $x \in \mathbb{R}$, is increasing, each of th etwo above equations has at most one root. One easily verifies that $r_1 = 3$ satisfies the firest equation and $r_2 = 4$ satisfies the second one. Thus $r_1 + r_2 = 7$.

Problem 9: *In $\triangle ABC$, produce a line from B to AC, meeting at D, and from C to AB, meeting at E. Let BD and CE meet at X. Let $\triangle BXE$ have area a, $\triangle BXC$ have area b, and $\triangle CXD$ have area c. Find the area of quadrilateral $AEXD$ in terms of a, b, and c.*



Solution: The triangle $\triangle BXE$ has area a, $\triangle BXC$ has area b, and $\triangle CXD$ has area c.

We will use the fact that the area of a triangle is equal to

$$1/2 \cdot \text{base} \cdot \text{ perpendicular height}.$$

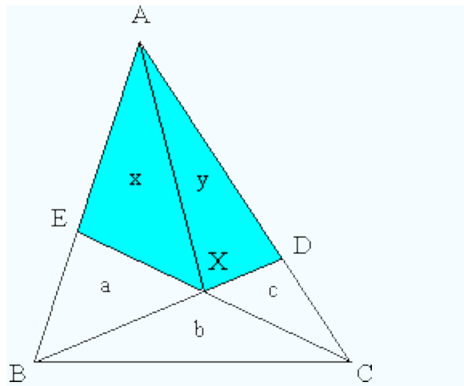
Any side can serve as the base, and then the perpendicular height extends from the vertex opposite the base to meet the base (or an extension of it) at right angles.

Consider triangles $\triangle BXE$ and $\triangle BXC$, with collinear bases EX and XC, respectively. The triangles have common height; therefore

$$\frac{|EX|}{|XC|} = \frac{a}{b}.$$

Similarly, considering triangle $\triangle BXC$ and $\triangle CXD$, with respective bases BX and XD,

$$\frac{|BX|}{|XD|} = \frac{b}{c}.$$



Now draw line AX. Let the triangle $\triangle AXE$ have area x and the triangle $\triangle AXD$ have area y. Consider $\triangle AXB$ and $\triangle AXD$, with bases BX and XD, such that $BX/XD = b/c$. Since the triangles $\triangle AXB$ and $\triangle AXD$ have common height, we have

$$\frac{a+x}{y} = \frac{b}{c}.$$

Similarly, considering $\triangle AXE$ and $\triangle AXC$, with collinear bases EX and XC,

$$\frac{x}{y+c} = \frac{a}{b}.$$

Hence,

$$by = cx + ac \quad \text{and} \quad bx = ay + ac.$$

Solving these simultaneous equations, we obtain

$$x = \frac{ac(a+b)}{b^2 - ac}, \qquad y = \frac{ac(b+c)}{b^2 - ac}.$$

Therefore the area of quadrilateral AEXD is

$$\frac{ac(a+2b+c)}{b^2 - ac}$$

Problem 10: *Find all integer solutions x, y, and z for the equation*

$$3^x + 4^y = 5^z$$

Solution: We will consider three separate cases: $x > 0$, $x = 0$, and $x < 0$.

CASE: $x > 0$
First of all, we note that

$$x > 0 \Rightarrow z > 0,$$

and then $y \geq 0$. Considering

$$3^x + 4^y = 5^z \mod 3,$$

we obtain

$$1 \equiv (1)^z \mod 3.$$

Hence z is even. Letting $z = 2w$, we may write 3^x as a difference of two squares:

$$3^x = 5^{2w} - 4^y = (5^w + 2^y)(5^w - 2y).$$

By the Fundamental Theorem of Arithmetic, each factor must be a power of 3, but, as their sum is not divisible by 3, both cannot be multiples of 3. Hence

$$5^w + 2^y = 3^x \quad \text{and} \quad 5^w - 2^y = 1.$$

Considering these equations, $\mod 3$, we get

$$(-1)^w + (-1)^y \equiv 0 \mod 3 \qquad \text{and} \qquad (-1)^w - (-1)^y \equiv 1 \mod 3.$$

Adding, we obtain

$$2 \cdot (-1)^w \equiv 1 \mod 3,$$

from which $(-1)^w \equiv -1 \mod 3$, and so w is odd.

Similarly, subtracting, we conclude that y is even. If $y > 2$, then, since w is odd, $5^w + 2^y \equiv 5 \mod 8$. However,

$$\text{either} \quad 3^x \equiv 1 \mod 8 \quad \text{or} \quad 3^x \equiv 3 \mod 8.$$

This is a contradiction; hence there is no solution with $x > 0$, $y > 2$.

If we assume $y = 2$, we have $5^w - 4 = 1$. Hence $w = 1$, and so $z = 2$. Then we must have $x = 2$, and $x = y = z = 2$ is a solution.

If we assume $y = 0$, then we have $3^x + 1 = 5^z$. Considering this equation $\mod 4$, we obtain $3^x \equiv 0 \mod 4$, which is impossible. Hence the only solution with $x > 0$ is $x = y = z = 2$.

CASE: $x = 0$
We have $1 + 4^y = 5^z$. Note that we must have $z > 0$, and so $y \geq 0$. By inspection, $y = z = 1$ is a solution.

Considering our equation $\mod 3$, we have $1 + 1 \equiv 2^z \mod 3$. Hence z is odd. Considering our equation $\mod 8$, if $y > 1$, we have $1 \equiv 5^z \mod 8$. Hence z is even. This is a contradiction; hence there is no solution with $x = 0$, $y > 1$. We conclude that the only solution with $x = 0$ has $y = z = 1$.

CASE: $x < 0$
Note that

$$x < 0 \ \wedge y \geq 0 \ \Rightarrow z > 0,$$

for which there is clearly no solution. So we must have $x < 0$ and $y < 0$, in which case $z < 0$. We may let $a = -x$, $b = -y$, $c = -z$, so that a, b, c are positive, and we have

$$\frac{1}{3^a} + \frac{1}{4^b} = \frac{1}{5^c}.$$

Multiplying throughout by $3^a 4^b 5^c$, we obtain

$$5^c(4^b + 3^a) = 3^a 4^b.$$

This is impossible as the right–hand side contains no factor of 5. We conclude that there is no solution with $x < 0$.

Conclusion: The only integer solutions are $(x, y, z) = (2, 2, 2)$ or $(0, 1, 1)$.