Solution to Problems ♠–6

Problem A: *Find the greatest lower and the least upper bounds of the set*

$$\left\{\frac{(n+1)^2}{2^n} : n \in \mathbb{N}\right\}.$$

Answer: First we are going to show the following three observations.

Claim A *For every real number $x \geq 4$ we have*

$$(x+1)^3 \leq 2x^3.$$

Proof of the Claim. Let $f(x) = (x+1)^3$ and $g(x) = 2x^3$ for $x \in \mathbb{R}$. Clearly $f'(x) = 3(x+1)^2$ and $g'(x) = 6x^2$. Also

(1) $f(4) = 125 < 128 = g(4)$, and
(2) $f'(x) < g'(x)$ for $x \geq 4 > 1 + \sqrt{2}$.

Therefore $f(x) < g(x)$ for all $x \geq 4$ and our Claim easily follows. □

Claim B *For every natural number $n \geq 11$ we have*

$$(n+1)^3 < 2^n.$$

Proof of the Claim. We show our Claim by induction on $n \geq 11$. Let $P(n)$ be the assertion that the inequality holds for n and let us verify that the assumptions of the Theorem on Mathematical induction are satisfied by the formula $P(n)$.
Basic Step $n = 11$
By direct computation we check that $(11+1)^3 = 1728 < 2048 = 2^{11}$, so $P(11)$ holds true indeed.
Inductive Step Let $n \geq 11$ and let us assume that $P(n)$ holds true, that is we assume

$(*)^n$ $(n+1)^3 < 2^n$.

We want to derive that then $P(n+1)$ is true. Using Claim A (for $x = n+1$)) and then $(*)^n$ we get

$$\Big((n+1)+1\Big)^3 \leq 2\cdot(n+1)^3 < 2\cdot 2^n = 2^{n+1}.$$

Consequently, $P(n+1)$ holds true.

Thus the assumptions of the Theorem on Mathematical Induction are satisfied and we may conclude that $(\forall n \geq 11)\ P(n)$, as desired. □

Claim C *For every natural number $n \geq 6$ we have*

$$(n+1)^2 < 2^n.$$

Proof of the Claim. We show our Claim by induction on $n \geq 6$. Let $P(n)$ be the assertion that the inequality holds for n and let us verify that the assumptions of the Theorem on Mathematical induction are satisfied by the formula $P(n)$.

Basic Step $n = 6$

By direct computation we check that $(6+1)^2 = 49 < 64 = 2^6$, so $P(6)$ holds true indeed.

Inductive Step Let $n \geq 6$ and let us assume that $P(n)$ holds true, that is we assume

$(**)^n\ \ (n+1)^2 < 2^n.$

We want to derive that then $P(n+1)$ is true. For this we note that for all $n \geq 6$ we have $2(n+2) < 2^n$. Now, using $(**)^n$, we get

$$\Big((n+1)+1\Big)^2 = (n+1)^2+2(n+1)+1 < 2^n+2(n+2) < 2^n+2^n = 2^{n+1}.$$

Consequently, $P(n+1)$ holds true.

Thus the assumptions of the Theorem on Mathematical Induction are satisfied and we may conclude that $(\forall n \geq 6)\ P(n)$, as desired. □

It follows from Claim B that

$$0 < \frac{(n+1)^2}{2^n} < \frac{(n+1)^2}{(n+1)^3} = \frac{1}{(n+1)} \quad \text{for all } n \geq 11.$$

Therefore 0 is the greatest lower bound of our set.

By Claim C we know that

$$\frac{(n+1)^2}{2^n} < 1 \quad \text{for all } n \geq 6.$$

The numbers $2, \frac{9}{4}, \frac{25}{16}, \frac{36}{32}$ (greater than 1) also belong to our set. Thus the least upper bound of the set is $\frac{9}{4}$.

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Problem B: *Show that for any irrational number α and for any positive integer n there exist a positive integer q_n and an integer p_n such that*

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{nq_n}.$$

Answer: Fix a natural number n and consider the $n+1$ real numbers

$$0,\ \alpha - \lfloor\alpha\rfloor,\ 2\alpha - \lfloor 2\alpha\rfloor,\ \ldots,\ n\alpha - \lfloor n\alpha\rfloor.$$

Since α is irrational, these numbers must be distinct. Each of these numbers belongs to the interval $[0,1)$. Since the n intervals $\left[\frac{j}{n}, \frac{j+1}{n}\right)$, $j = 0, 1, \ldots, n-1$ cover $[0,1)$, there must be one which contains at least two of these points, say $n_1\alpha - \lfloor n_1\alpha\rfloor$ and $n_2\alpha - \lfloor n_2\alpha\rfloor$ with $0 \le n_1 < n_2 \le n$. So

$$\left|n_2\alpha - \lfloor n_2\alpha\rfloor - n_1\alpha + \lfloor n_1\alpha\rfloor\right| < \frac{1}{n}$$

and dividing both sides of the inequality by $n_2 - n_1 > 0$ we get

$$\left|\alpha - \frac{\lfloor n_2\alpha\rfloor - \lfloor n_1\alpha\rfloor}{n_2 - n_1}\right| < \frac{1}{n(n_2 - n_1)}.$$

Thus it is enough to take $q_n = n_2 - n_1$ and $p_n = \lfloor n_2\alpha\rfloor - \lfloor n_1\alpha\rfloor$.

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