Solution to Problems ♡–5

Problem A: *Show that every number in the sequence*

$$1007, \quad 10017, \quad 100117, \quad 1001117, \quad 10011117, \ldots$$

is divisible by 53.

Answer: For a natural number n let a_n be the natural number with the decimal expansion $100\underbrace{1\ldots1}_{n-1}7$. One should notice that

$$a_{n+1} = (a_n - 6) \cdot 10 + 7 = 10a_n - 53.$$

Now by a straightforward induction on $n \in \mathbb{N}$ we show that each number a_n is divisible by 53. To this end we verify that the assumptions of the Theorem on Mathematical Induction are satisfied.

Basic Step: $n = 1$
We note that $a_1 = 1007 = 19 \cdot 53$, so our claim is readily true for $n = 1$.
Inductive Step: Let $n \in \mathbb{N}$ be an arbitrary natural number and let us assume that

$(\circledast)_0$ a_n is divisible by 53.

Thus for some integer k we have

$(\circledast)_1$ $a_n = 53 \cdot k$.

Now,

$$a_{n+1} = 10a_n - 53 = 10 \cdot 53 \cdot k - 53 = 53 \cdot (10k - 1).$$

Since $10k - 1$ is an integer, we conclude that (under our inductive assumption $(\circledast)_0$), a_{n+1} is divisible by 53. Thus if our claim is true for n, then it is also true for $n + 1$.

Consequently, by the Theorem on Mathematical Induction we may conclude that $(\forall n \in \mathbb{N})(53 \mid a_n)$.

Correct solutions were received from :

(1) Ali Al Kadhim — POW 5A: ♡
(2) Cody Anderson — POW 5A: ♡
(3) Gage Hoefer — POW 5A: ♡
(4) Sarah McCarty — POW 5A: ♡
(5) Mackenzie McClure — POW 5A: ♡
(6) Bryce W. Sahs — POW 5A: ♡
(7) Worthy S. Tudent — POW 5A: ♡
(8) Brad Tuttle — POW 5A: ♡

Problem B: *Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.*

Answer: For a natural number $n \geq 8$ let $P(n)$ be the assertion that

> a square can be dissected into k smaller squares (not necessarily all of the same size) for $k = n$ and for $k = n-1$ and for $k = n-2$.

It should be clear that the sentence

$$(\diamondsuit) \qquad (\forall n \geq 8)\ P(n)$$

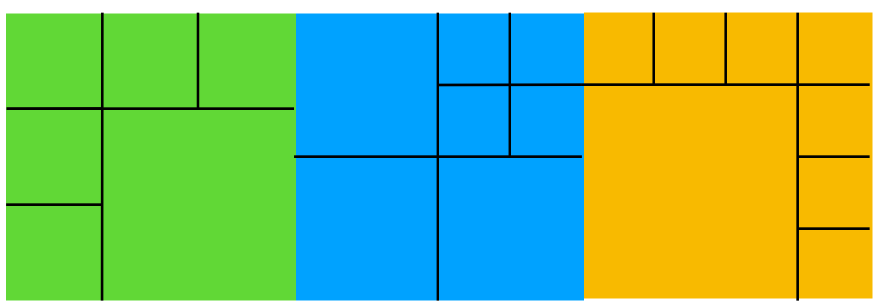
is equivalent to the claim that

> for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.

We will show ($\diamondsuit$) using the Theorem on Mathematical Induction. To this end we will verify that the formula $P(n)$ satisfies the assumptions of this theorem.

Basic Step: $n = 8$

We have to justify that a square can be partioned into 8, 7, and 6 squares (not necessarily all of the same size). But this follows by the following pictures.



Inductive Step: Let $n \geq 8$ be an arbitrary natural number and let us assume that $P(n)$ holds true, that is

($\boxdot$) a square can be dissected into k smaller squares (not necessarily all of the same size) for $k = n$ and for $k = n-1$ and for $k = n-2$.

We are going to argue that then $P(n+1)$ holds true. First we note that by our assumption ($\boxdot$), a square can be divided into $n = (n+1)-1$ and into $n-1 = (n+1)-2$ squres. To create a partition into $n+1$ squares we use the the inductive hypothesis ($\boxdot$) to divide a square into

$k = n - 2$ squares. Take one of these $n - 2$ squares and divide it into four identical smaller squares. This increases the number of subsquares by 3, producing a square broken up into $(n - 2) + 3 = n + 1$ squares. Thus if our claim is true for n, then it is also true for $n + 1$.

Consequently, by the Theorem on Mathematical Induction we may conclude that $(\diamondsuit)$ holds true.

Correct solutions were received from :

(1) Ali Al Kadhim POW 5B: $\heartsuit$
(2) Gage Hoefer POW 5B: $\heartsuit$
(3) Worthy S. Tudent POW 5B: $\heartsuit$
(4) Brad Tuttle POW 5B: $\heartsuit$