Solution to Problems ♠–4

Problem A: *Let $m > 1$ be a natural number. Prove that there exist an integer $k > 1$ such that*

- *k is divisible by m, and*
- *the decimal expansion of k has zeros and ones only.*

Answer: For a natural number $n \geq 1$ let

$$a_n = \underbrace{1\dots1}_{n} = \sum_{i=0}^{n-1} 10^i.$$

In the infinite sequence $\left(a_n\right)_{n=1}^{\infty}$, there must be two terms congruent modulo m (i.e., giving the same remainder when divided by m). Say, $n_1 < n_2$ are such that

$$a_{n_1} \equiv a_{n_2} \mod m.$$

Then m divides $a_{n_2} - a_{n_1}$ and the decimal expansion of $k = a_{n_2} - a_{n_1}$ has the form

$$k = \underbrace{1\dots1}_{n_2-n_1}\underbrace{0\dots0}_{n_1} = \sum_{i=n_1}^{n_2-1} 10^i.$$

Correct solution was received from :

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Problem B: *Let $a, b, x_0 \geq 1$ be natural numbers and for $n \geq 1$ let*

$$x_n = ax_{n-1} + b.$$

Show that there are infinitely many composite numbers in the sequence $\big(x_n\big)_{n=1}^{\infty}$.

Answer: Plainly, the sequence $\big(x_n\big)_{n=1}^{\infty}$ is strictly increasing.

If $d \geq 2$ divides a and b, then d divides all terms x_n of our sequence, so all x_n for $n > 1$ are composite.

Thus we may assume that the integers a, b are relatively prime. Then also a and x_n are relatively prime (for each $n \geq 1$).

Fix k for a moment and consider the following $x_k + 1$ terms of our sequence':

$$x_k, x_{k+1}, \ldots, x_{k+x_k}.$$

Two of these terms must be congruent modulo x_k (i.e., they give the same remainder when divided by x_k). Say, $k \leq p < q \leq k + x_k$ are such that

$$x_p \equiv x_q \mod x_k.$$

Then x_k divides $x_q - x_p$. But $x_q - x_p = a(x_{q-1} - x_{p-1})$ and x_k, a are relatively prime, so x_k must divide $x_{q-1} - x_{p-1}$. Since $x_{q-1} - x_{p-1} = a(x_{q-2} - x_{p-2})$, we conclude that x_k divides $x_{q-2} - x_{p-2}$. Continuing in this fashion for $p - k$ steps we will eventually obtain that x_k divides $x_{q-(p-k)} - x_k$, so x_k divides $x_{k+q-p} > x_k$. Consequently, x_{k+q-p} is a composite number.

The argument above shows that *for every* k there is an $n > k$ such that x_n is composite. Therefore there are indeed infinitely many composite numbers in our sequence.

No correct solutions received