Solution to Problems ♠–3

Problem A: *Prove that for any integer m the number $\frac{m}{3}+\frac{m^2}{2}+\frac{m^3}{6}$ is an integer.*

Answer: For any integer number m

$$\frac{m}{3}+\frac{m^2}{2}+\frac{m^3}{6}=\frac{m(m+1)(m+2)}{6}$$

One of two sequential integers m and $m+1$ is divisible by 2, so 2 divides $m(m+1)$, and one of three sequential integers $m, m+1$, and $m+2$ is divisible by 3, so 3 divides $m(m+1)(m+2)$. Consequently, $2\cdot 3=6$ divides $m(m+1)(m+2)$.

Thus

$$\frac{m(m+1)(m+2)}{6}=\frac{m}{3}+\frac{m^2}{2}+\frac{m^3}{6}$$

is an integer number.

Correct solution was received from :

(1) Mackenzie McClure — POW 3A: ♠

(2) Brad Tuttle — POW 3A: ♠

Problem B: *Evaluate the integral*

$$\int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} dx.$$

Answer: Solution:

Let $x = \pi/2 - y$ and $R = \int_0^{\pi/2} \frac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x} dx$, and $\alpha = 1/3$. Then

$$R = \int_{\pi/2}^{0} \frac{\sin^\alpha(\frac{\pi}{2} - y)}{\sin^\alpha(\frac{\pi}{2} - y) + \cos^\alpha(\frac{\pi}{2} - y)}(-dy)$$

$$= \int_0^{\pi/2} \frac{\cos^\alpha y}{\cos^\alpha y + \sin^\alpha y} dy$$

$$= \int_0^{\pi/2} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx$$

Therefore

$$\int_0^{\pi/2} \frac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x} dx + \int_0^{\pi/2} \frac{\cos^\alpha x}{\cos^\alpha x + \sin^\alpha x} dx = 2R$$

or $2R = \int_0^{\pi/2} dx$. Consequently, $R = \frac{1}{2} \int_0^{\pi/2} dx = \frac{\pi}{4}$.

Correct solution was received from :

(1) Ali Al Kadhim — POW 3B: ♠

(2) Brad Tuttle — POW 3B: ♠