

## Solution to Problems ♠–2

**Problem A:** *Five professors attended a lecture. Each fell asleep just twice. For each pair there was a moment when both were asleep. Show that there was a moment when three of them were asleep.*

**Answer:** For each pair take the first moment when they are both asleep. There are ten pairs, so there are ten such moments. If two of these moments coincide, then we are done because at that moment at least three professors were asleep.

So suppose towards contradiction that they are all distinct and form a set  $S$ . Each moment in  $S$  must also be one of the 10 occasions when a professor falls asleep (as each professor fell asleep just twice.) But consider the earliest member of  $S$ . Two professors were asleep at that moment so two fell asleep at or before that moment. Thus each of the remaining 9 members of  $S$  must be one of the 8 later occasions when a professor fell asleep. But they cannot all be distinct, a contradiction.

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(1) BRAD TUTTLE

POW 2A: ♠

**Problem B:** *A license plate has six digits from 0 to 9 and may have leading zeros. If two plates must always differ in at least two places, what is the largest number of plates that is possible? .*

**Answer:**  $10^5$ . To justify this, we first show by induction on  $n$  that we can find a set of  $10^{n-1}$  plates for  $n > 1$  digits. Let us verify the assumptions of Theorem on Mathematical Induction.

**Basic Step:** Our claim is true for  $n = 2$ .

Why? Just consider the plates 00, 11, 22,  $\dots$ , 99.

**Inductive Step:** Assume that our claim is true for  $n \geq 2$  and let us argue that then it is true for  $n + 1$  too.

Let  $S$  be a set of  $10^{n-1}$  plates with  $n$  digits and such that any two plates in  $S$  differ in at least two places.

If  $d$  is a digit from 0 to 9 and  $s$  is a plate of  $n$  digits, let  $[d, s]$  be the plate of  $n + 1$  digits which has  $d$  as its first digit, and the remaining digits the same as those of  $s$ , except that the last digit is that for  $s$  plus  $d$  (reduced mod 10 if necessary). Let

$$S' = \{[d, s] : d = 0, 1, \dots, 9 \text{ and } s \text{ belongs to } S\}$$

The set  $S'$  obviously has 10 times as many members as  $S$ .

We claim that any two plates in  $S'$  differ in at least two places, that is  $[a, s]$  and  $[b, t]$  differ in at least two places when either  $a \neq b$  or  $s \neq t$ . We consider three possible cases.

- If  $a = b$  and  $s \neq t$ , then  $s$  and  $t$  differ in at least two places. The same change is made to their last digits, so  $[a, s]$  and  $[a, t]$  differ in at least two places.
- If  $a \neq b$  and  $s = t$ , then  $[a, s]$  and  $[b, s]$  differ in both their first and last places.
- If  $a \neq b$  and  $s \neq t$ , then  $s$  and  $t$  differ in at least two places and so the modified  $s$  and  $t$ , differ in at least one place. But  $[a, s]$  and  $[b, t]$  also differ in the first place, so they differ in at least two places.

The inductive step is verified.

Thus, by Theorem on Mathematical Induction, our claim is true for any  $n \geq 2$ . So we have established that the largest number is at least  $10^{n-1}$  for  $n$  digits.

But any two plates which differ only in the last digit cannot both be chosen. So at most 1/10 of the  $10^n$  possible plates can be chosen. That shows that  $10^{n-1}$  is best possible.

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