Solution to Problems ♠–12

Problem A: *There are 9 delegates at a conference, each speaking at most three languages. Given any three delegates, at least 2 speak a common language. Show that there are three delegates with a common language.*

Answer: Suppose towards contradiction that no three delegates speak the same language.

Then every candidate can share a language with at most 3 other delegates, because if (s)he shared a language with 4, (s)he would have to share the same language with 2 of them (since (s)he can only speak 3 languages). This creates a triple of delegates each speaking that shared language.

Let A be one of the delegates. By the preceding paragraph, there are 5 delegates who do not share a language with A. Let one of them be B. By the same argument, there must be at least one of the other 4 (call her C) who does not share a language with B. But now no two of A, B, C share a language, a contradiction.

Correct solutions were received from :

(1) Brad Tuttle

Problem B: *A set X has n elements, $n \geq 3$. Given $n+1$ subsets of X, each with 3 members, show that we can always find two which have just one element in common.*

Answer: We show this claim by induction on $n \geq 3$.

Let $P(n)$ be the assertion that

> **if** *a set $X \neq \emptyset$ has at most n elements, and $\mathcal{A}$ is a collection of at least $|X|+1$ subsets of X, each with 3 members,*
> **then** *there are $A, B \in \mathcal{A}$ such that $|A \cap B| = 1$.*

We will verify that the formula $P(n)$ satisfies the assumptions of the Theorem on Mathematical Induction.

[**Basic Step**]: We note that $P(3)$ asserts that

> if $|X| \leq 3$ and $\mathcal{A}$ is a collection of $|X|+1$ subsets of X, each with 3 members,
> then there are $A, B \in \mathcal{A}$ such that $|A \cap B| = 1$.

However, a set with at most 3 elements has at most one 3 element subset, so the hypothesis of the above implication cannot be satisfied. Consequently whole implication is satisfied vacuously, and the statement $P(3)$ is true.

[**Inductive Step**]: We are going to show that

$$\big(\forall n \geq 3\big)\big(P(n) \ \Rightarrow \ P(n+1)\big).$$

To this end suppose that $n \geq 3$ is arbitrary but fixed. Assume also that $P(n)$ holds true, that is

$(\oplus)_n$ if $|X| \leq n$ and $\mathcal{A}$ is a collection of at least $|X|+1$ subsets of X, each with 3 members, then there are $A, B \in \mathcal{A}$ such that $|A \cap B| = 1$.

Assume that Y is a set with at most $n+1$ elements and $\mathcal{B}$ is a collection of at least $|Y|+1$ subsets of X, each with 3 members. If $|Y| \leq n$ then our inductive assumption $(\oplus)_n$ applies to Y and the desired conclusion follows. So we may assume $|Y| = n+1$ and $|\mathcal{B}| = n+2$.

Suppose towards contradiction that

$(\circledast)$ $\big(\forall A, B \in \mathcal{B}\big)\big(|A \cap B| \neq 1\big)$.

If every element of Y was in at most 3 of the sets from $\mathcal{B}$, there would be at most $n+1$ subsets, so some $a \in Y$ is in at least 4 of the subsets from $\mathcal{B}$. Suppose one of them is $A = \{a, b, c\} \in \mathcal{B}$. There are at least three others sets $B, C, D \in \mathcal{B}$ containing a, and each of them must intersect $\{b, c\}$ (because of $(\circledast)$). Consequently, b (say) must be in at least two of B, C, D.

Without loss of generality, $B = \{a, b, d\}$ and $C = \{a, b, e\}$. By our assumption (⊛), any other set $I \in \mathcal{B}$ containing a must intersect each of the sets $A \setminus \{a\}$, $B \setminus \{a\}$ and $C \setminus \{a\}$. Therefore,

$(*)_1$ every set $I \in \mathcal{B}$ containing a must contain b,

because otherwise it would have to contain c, d and e, which is impossible. Similarly,

$(*)_2$ every set $I \in \mathcal{B}$ containing b must contain a.

Thus

$$\mathcal{B}^* \stackrel{\text{def}}{=} \{E \in \mathcal{B} : \{a, b\} \cap E \neq \emptyset\} = \{E \in \mathcal{B} : a, b \in E\}.$$

Let $m = |\mathcal{B}^*|$ and note that $m + 2 \leq n + 1$, so $m \leq n - 1$.

If $\{a, b, k\} \in \mathcal{B}^*$, then k cannot belong to any other set $E \in \mathcal{B}$: if $k \in E$, then (⊛) implies $\{a, b\} \cap E \neq \emptyset$ and consequently also $a, b \in E$, so $E = \{a, b, k\}$.

Consider the set X of the $(n + 1) - (m + 2) = n - m - 1$ elements other than those which belong to sets in $\mathcal{B}^*$, i.e.,

$$X = Y \setminus \bigcup \mathcal{B}^*.$$

Let $\mathcal{A} = \mathcal{B} \setminus \mathcal{B}^*$. Then for each $E \in \mathcal{A}$ we have $E \subseteq X$. Also, $|\mathcal{A}| = |\mathcal{B}| - |\mathcal{B}^*| = n - m + 2 \geq n - (n - 1) + 2 = 3$. Consequently $|X| \geq 4$ and $n + 1 \geq 4 + (m + 2)$, so $|\mathcal{B}| = n + 2 \geq m + 7$. This gives $|\mathcal{A}| \geq 7$ and hence $5 \leq |X| \leq n$ and $|X| + 1 \leq |\mathcal{A}|$. Applying the inductive hypothesis $(\oplus)_n$ to X and $\mathcal{A}$ we find sets $A, B \in \mathcal{A} \subseteq \mathcal{B}$ such that $|A \cap B| = 1$, contradicting (⊛).

Therefore $P(n + 1)$ is true. Thus we have shown that

$$P(n) \;\Rightarrow\; P(n + 1)$$

and as our n was arbitrary we may conclude

$$\big(\forall n \geq 3\big)\big(P(n) \;\Rightarrow\; P(n + 1)\big).$$

Consequently the assumptions of the Theorem on Mathematical Induction are satisfied and, by this theorem, we may conclude that the claim in the problem holds true.

No Correct Solutions Received